1970

The Mhd Rayleigh Problem With Unsteady Wall Motion.

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70-18,556

SHIH, Pei-Chong, 1939-
THE MHD RAYLEIGH PROBLEM WITH UNSTEADY WALL MOTION.

The Louisiana State University and Agricultural and Mechanical College, Ph.D., 1970
Engineering Mechanics

University Microfilms, Inc., Ann Arbor, Michigan

THE MHD RAYLEIGH PROBLEM WITH UNSTEADY WALL MOTION

A Thesis

Submitted to the Graduate Faculty of the

Louisiana State University and

Agricultural and Mechanical College

in partial fulfillment of the

requirements for the degree of

Doctor of Philosophy

in

The Department of Engineering Science

by

Pei-Chong Shih

M.S., Louisiana State University, 1966

January, 1970

Acknowledgments

I want to express my sincere gratitude to Dr. D. W. Yannitell, chairman of my special committee and thesis advisor. His continuing guidance and encouragement have made this thesis possible and I am proud to have worked with him.

Sincere thanks are due Professor D. R. Carver, head of the Department of Engineering Science, for his financial support to make my studying in this department possible.

Shih, Pei-Chong, B.S., Provincial Cheng-Kung University, 1962; M.S.C.E., Louisiana State University, 1966

Doctor of Philosophy, January, 1970

Major: Engineering Mechanics; Minor: Mathematics

<u>The MHD Rayleigh Problem with Unsteady Wall Motion</u>

Dissertation directed by Dr. D. W. Yannitell

Pages in dissertation, [illegible]. Words in Abstract, 409.

The flow of an electrically conducting, viscous, incompressible fluid, due to the decelerating impulsive motion as well as accelerating motion of an infinite flat wall, is first discussed. A transverse, uniform, fixed-in-space magnetic field is supposed to be present. Induced magnetic field is retained in the governing equations given by Ludford (1959) and nondimensional variables introduced by Steketee (1964) are applied. Laplace transform techniques are used.

The wall velocity is assumed to be $(I(t)U_* \exp(-wt))$ in the case of decelerating impulsive wall motion and $(I(t)U_* \{1 - \exp(-wt)\})$ in the case of accelerating wall motion. $I(t)$ is the unit step function at $t = 0$, w and U_* are constants and t is the time. The classical Rayleigh problem and MHD Rayleigh problem are two special cases of the problem of decelerating wall motion. The solution for the accelerating wall motion follows immediately from that for the decelerating impulsive wall motion by superposition.

The solution, in general, is very complicated and can not be expressed in terms of known functions. However, the solutions in certain cases have been found for $((\eta^{\frac{1}{2}} + \nu^{\frac{1}{2}})^2 w/a^2 \leq 1)$ even though the corresponding solution does not exist in ordinary fluid mechanics. a is the Alfvén velocity, η and ν are the magnetic diffusivity and kinematic viscosity of the fluid respectively.

For the decelerating impulsive wall motion, it is observed that, in the case of equal diffusion coefficients, the shear stress at the interface and the disturbances in both the velocity and magnetic field increase with an increase in the wall conductivity or the applied magnetic field, and with a decrease in w. The speed of the diffusing Alfvén wave is modified to $(a^2 - 4\eta w)^{\frac{1}{2}}$, leaving the solution dominated by the effect of the viscous boundary layer diffusing from the wall. The induced magnetic field is, in general, not negligible.

For the accelerating wall motion, it is found that, in the case of equal diffusion coefficients, the shear stress at the interface and the disturbances in both the velocity and magnetic field increase as time advances with the increase in the wall conductivity or the value of w and with the decrease in the applied magnetic field. The induced magnetic field is not negligible except for small time.

The concept of the characteristic signal lines, together with the exact solution previously obtained, is then applied and a general transient solution for a non-dissipative fluid is obtained for the arbitrary motion of a perfectly conducting or non-conducting wall.

LIST OF ILLUSTRATIONS

LIST OF NOMENCLATURE

The following notation, unless otherwise defined, is employed:

a Alfvén velocity, $a = B_o/(\rho\mu)^{\frac{1}{2}}$

b, B nondimensional and dimensional x-component of the magnetic field intensity vector $\underline{B}$

$\underline{B}$ magnetic field intensity vector

B_o applied magnetic field intensity

c constant

C_1, C_2, C_3 coefficients in Laplace transform

C_f coefficient of the shear stress at the interface

$D_{\pm}$ linear differential operator

E Cartesian z-component of the electric field vector $\underline{E}$

$\underline{E}$ electric field vector

F total shear stress at the interface

j Cartesian z-component of the current vector $\underline{J}$

$\underline{J}$ current vector

k constant

$l_{\pm}$ characteristic signal lines

m, n constants

Pr_m magnetic Prandtl number, $Pr_m = (\rho\sigma\nu)^{\frac{1}{2}}$

s Laplace transform variable

t time

u, U nondimensional and dimensional x-component of the velocity vector $\underline{V}$ respectively, u = U/a

u_*, U_* constants, $u_* = U_*/a$

u_{wall}, U_{wall}, U_w wall velocity, $u_{wall} = U_{wall}/a$

$\underline{V}$ velocity vector

w constant

x, y, z Cartesian coordinates

α, β, γ fluid properties, $\alpha^{\frac{1}{2}} = \frac{1}{2}(\eta^{\frac{1}{2}} + \nu^{\frac{1}{2}})$, $\beta^{\frac{1}{2}} = \frac{1}{2}(\eta^{\frac{1}{2}} - \nu^{\frac{1}{2}})$,
$\gamma = (\beta/\alpha)^{\frac{1}{2}}$ ($\beta = (\rho\mu)^{-\frac{1}{2}}$ in Chapt. V)

σ electrical conductivity

η magnetic diffusivity, $\eta = 1/\mu\sigma$

ν kinematic viscosity

ρ density

μ magnetic permeability

ϵ square root of the ratio of diffusivities, $\epsilon = (\nu/\eta)^{\frac{1}{2}}$

τ, ξ nondimensional variables, $\tau = a^2t/\alpha$, $\xi = ay/\alpha$

Ω nondimensional parameter, $\Omega = \alpha w/a^2$

The subscript, s, refers to the solid; a bar above a variable refers to the Laplace transform of the variable.

CONTENTS

I. INTRODUCTION

A. The MHD Rayleigh Problem and Related Problems

1. Small perturbation in the induced magnetic field

The Rayleigh problem, or Stokes' first problem [1], studies the non-steady parallel flow of an infinitely extended, viscous, incompressible fluid near an infinite flat plate which is suddenly started from rest and moves in its own plane with a constant velocity U_*. The exact solution for the velocity distribution is

$$U = U_* \operatorname{erfc}(y/2\sqrt{\nu t}) \qquad \text{at } y \geq 0,\ t \geq 0, \tag{1.1}$$

where U is the velocity, ν is the kinematic viscosity of the fluid, t is the time and y is the perpendicular distance from the flat plate. The complementary error function of x, erfc(x), is defined as

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$$
$$= \frac{2}{\pi^{\frac{1}{2}}} \int_x^{\infty} \exp(-x^2)dx$$

which has been tabulated.

Stokes' second problem [1], an extension of the classical Rayleigh problem, discusses the flow of an infinitely extended, viscous, incompressible fluid near an infinite flat plate which executes simple harmonic oscillations in its own plane. A quasi-steady solution, in which the transient flow is neglected, can be expressed in terms of known functions. However, non-steady solution consists of the quasi-steady solution plus a integral which decays with increasing time.

The extension of the classical Rayleigh problem to magnetohydrodynamics was first attacked by Rossow [2]. Rossow studied the flow of an electrically conducting, viscous, incompressible fluid in the presence of a transverse uniform magnetic field B_0, past an infinitely extended flat plate moving impulsively from rest with a constant velocity. Ong and Nicholls [7] extended Rossow's problem to the case of an infinite flat plate executing simple harmonic motion in its plane. The induced magnetic field produced by the current was assumed to be a perturbation to the applied magnetic field and hence was neglected in both problems above.

It is worthwhile to examine Ong and Nicholls' solution obtained by Laplace transform methods. The solution, in the case of a magnetic field fixed relative to the fluid, is

$$U(y,t) = U_0 \exp(-k_+ y) \cos(nt - k_- y),$$

where

$$k_\pm = \left\{(1/2\nu)((m^2 + n^2)^{\frac{1}{2}} \pm m)\right\}^{\frac{1}{2}}$$

and $$m = \sigma B_0^2/\rho.$$

This solution indeed satisfies the approximate governing equation with the induced magnetic field neglected;

$$\frac{\partial U}{\partial t} + mU = \frac{\partial^2 U}{\partial y^2} \quad ,$$

and the boundary conditions,

$$U = U_0 \cos(nt) \quad \text{at } y = 0,\ t \geq 0,$$

$$U = \text{finite} \quad \text{at } y = \infty,\ t \geq 0.$$

However, the assumed initial condition: $U = 0$ at $t < 0$, is not satisfied and can be easily checked by setting $t < 0$ in the solution. Unfortunately, this initial condition is necessary in order to apply Laplace transform techniques in this problem. This shows that the solution obtained by means of the inversion theorem is not exact, but it can be considered as the quasi-steady solution. This is more clear in the solution of the special case, $m = 0$,

$$U(y,t) = U_0 \exp(-(n/2\nu)^{\frac{1}{2}} y) \cos(nt - (n/2\nu)^{\frac{1}{2}} y) \quad ,$$

which is identical to the quasi-steady solution of an oscillating plate in classical fluid mechanics [1].

Gupta [13] discussed the motion of an electrically conducting, viscous, incompressible fluid near an accelerating plate under a transversely applied magnetic field. Soundalgekan [14] investigated the flow of an electrically conducting, viscous, incompressible fluid near an accelerating plate in the presence of a parallel plate under transverse magnetic field. In these two problems, the induced magnetic field was also neglected in comparison with the applied field.

2. Large perturbations observed

The assumption of small disturbance in the induced magnetic field for certain problems in magnetohydrodynamics met with severe criticism. In classical fluid mechanics, the small disturbance theory has been used in the thin airfoil problem. However, the order-one disturbance was found in the aligned-fields flow of a perfectly conducting, inviscid, incompressible fluid past a thin airfoil in the sub-Alfvénic case considered by Leibovich and Ludford [15] and in the super-Alfvénic case by Ludford and Yannitell [16].

Large perturbations in the induced magnetic field have also been observed by Chang and Yen [3] and Ludford [4]. They reexamined the MHD Rayleigh problem in the case of a perfectly conducting flat plate moving impulsively from rest. Since the fluid motion is affected by the electrical conductivity of the plate, the induced magnetic field may not be considered a small perturbation to the applied magnetic field in many cases, and was retained in their analyses. The exact solution in terms of known functions is given for certain cases.

The effect of the electrical conductivity of the flat plate in the MHD Rayleigh problem has been investigated by Drake [5] and Bryson and Rosciszewski [6]. Axford [8] studied the flow of an electrically conducting, viscous, incompressible fluid near an infinite flat plate which executes simple harmonic oscillations in its plane in the case of a perfect conductor and Hide and Roberts [9] in the case of an insulator. The effect of the electrical conductivity of an infinite flat plate executing simple harmonic oscillations in its own plane has been investigated by Young and Hughes [10]. The induced magnetic field was retained in these problems. However, as in classical fluid mechanics (Stokes' second problem), only the quasi-steady solution is given, and not the transient solution.

3. The viscous boundary layer and Alfvén wave

The classical Rayleigh problem contains a viscous boundary-layer adjacent to the solid-fluid interface. With the introduction of electromagnetic effects, the significant features of the flow, as obtained by previous investigators mentioned above, are the formation of a viscous layer diffusing from the solid-fluid interface and the generation of an Alfvén wave which propagates into the fluid, at Alfvén speed, and diffuses. As the wave moves out of the viscous layer, the latter is fully developed and the flow behind the wave is quasi-steady. For very small dissipation, the changes in the velocity and magnetic field all take place in a thin boundary layer adjacent to the wall in the manner predicted by Hartmann for MHD channel flow.

In the non-dissipative limit ($\nu \to 0, \sigma \to \infty$), the viscous layer and the range of the diffusing Alfvén wave both shrink to a line and the limiting values of velocity and magnetic field are not independent. The jump in tangential velocity and magnetic field across the Alfvén wave are related by fluid properties;

$$[\underline{B}]_s = \mp \sqrt{\rho\mu}\,[\underline{V}]_s \,, \tag{1.2}$$

where $[\ ]_s$ denotes the jump in the tangential component of the argument, $\underline{B}$ is the magnetic intensity, $\underline{V}$ is the velocity, ρ is the density of the fluid and μ is the magnetic permeability of the fluid. The sign depends on whether the wave propagates in the direction of positive or negative magnetic field. The jumps in tangential velocity and magnetic field across the viscous layer are related by the magnetic Prandtl number Pr_m;

$$[\underline{B}]_s = \pm \mu\, Pr_m\, [\underline{V}]_s \,, \tag{1.3}$$

as given by Stewartson [12]. The sign is chosen to agree with solid to fluid direction. Stewartson's solution involves that of the one dimensional steady state MHD Rayleigh problem. Note that eq.(1.3) is valid only for a solid insulator with the presence of a normal component of magnetic field, and is applied as the solid-fluid boundary condition for a fluid with small dissipation. Under conditions(1.2) and (1.3), an arbitrary initial discontinuity at the solid-fluid interface is resolved into an Alfvén wave, which is emitted from the interface and propagates along the magnetic field into the fluid at Alfvén speed, and a Hartmann layer left at the interface. This result has been shown by Bryson and Rościszewski in the MHD Rayleigh problem.

B. Transient Solution for the Non-uniform Wall Motion

1. Non-steady wall motion

The object of this thesis is to find the first truly transient solutions for the non-steady motion of an electrically conducting wall other than that of the MHD Rayleigh problem in which the time derivative of the wall velocity vanishes.

We shall consider a viscous, electrically conducting, incompressible fluid occupying the half space $y > 0$, and a semi-infinite flat wall with electrical conductivity σ_s in the remaining space $y < 0$, with a transverse, uniform, fixed -in-space magnetic field applied as shown in fig.(1). Two types of wall motion are considered:

(i) decelerating impulsive wall motion -- the velocity of the wall, in the x-direction, is assumed to be

$$U_{wall} = I(t)U_{*}\exp(-wt) \ , \qquad (1.4)$$

where $I(t)$ is the unit step function at $t = 0$, t is the time, U_{*} and w are constants. Eq.(1.4) is plotted in fig.(2).

(ii) accelerating wall motion -- the velocity of the wall is assumed to be

$$U_{wall} = I(t)U_{*}\{1 - \exp(-wt)\} \ , \qquad (1.5)$$

which is plotted in fig.(3).

The first problem can be simplified to the MHD Rayleigh problem by setting $w = 0$, i.e., the wall would be im-

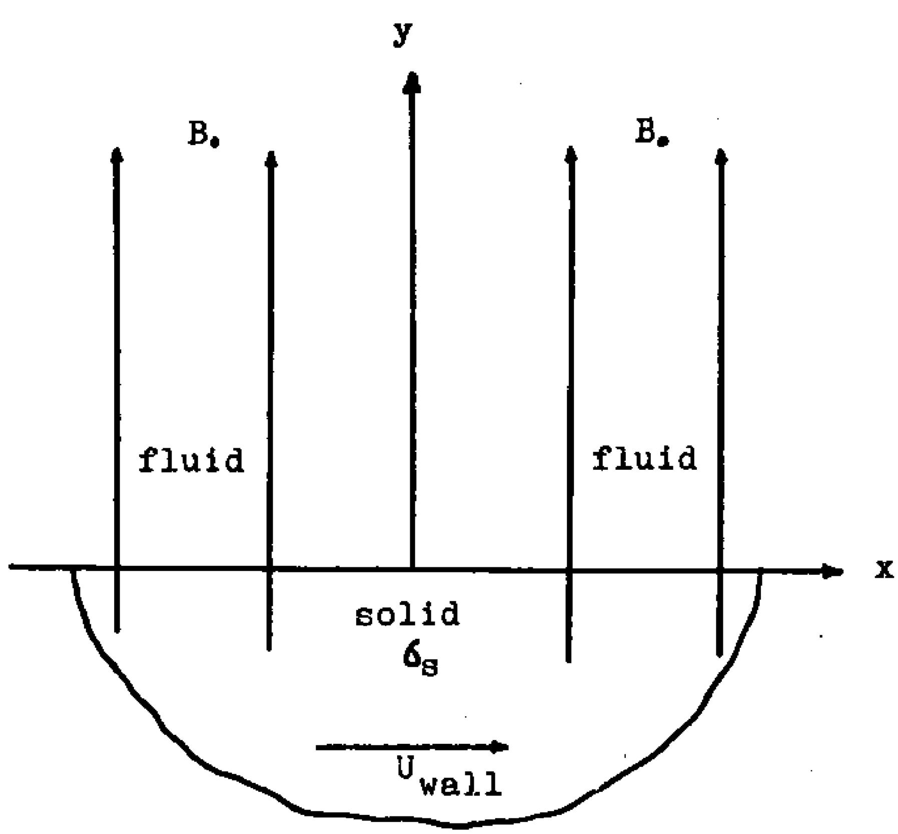

Fig. 1. Geometry of the problems considered.

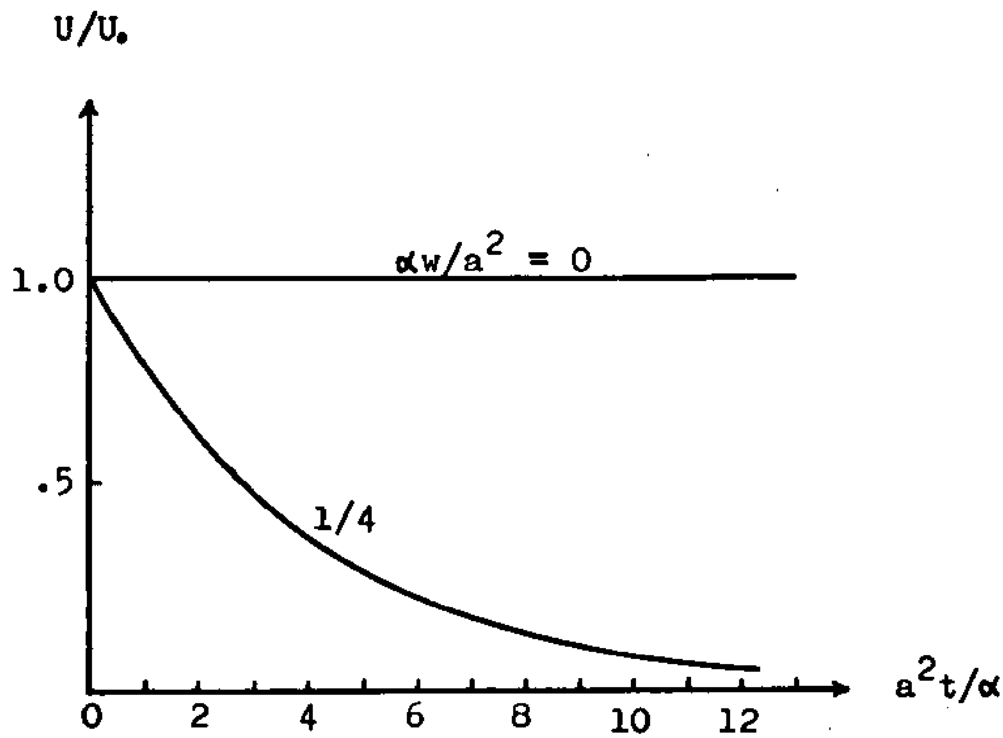

Fig. 2. Wall velocity: decelerating impulsive motion.

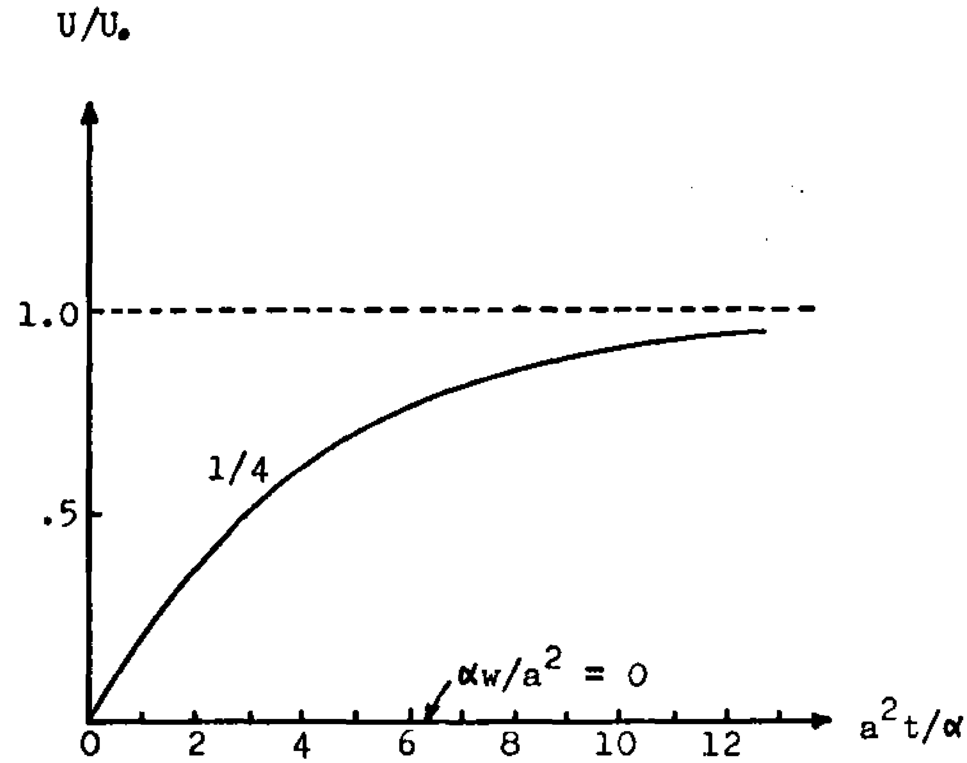

Fig. 3. Wall velocity: accelerating motion.

pulsively started from rest with a constant velocity U. in the x-direction.

2. Results for decelerating impulsive wall motion

As mentioned above, transient solutions have been obtained for the classical Rayleigh problem and the MHD Rayleigh problem. However only quasi-steady solutions have been found for the oscillating problems in magnetohydrodynamics as well as in classical fluid mechanics. The general solution of the decelerating impulsive motion of a conducting wall for arbitrary values of kinemetic viscosity ν, magnetic diffusivity η and the parameter w is, in general, very complicated and can not be expressed in terms of known functions as in the case of the MHD Rayleigh problem. Fortunately, some exact solutions have been found for

$$(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2 w \leq a^2 \ , \qquad (1.6)$$

even though the corresponding solution does not exist in classical fluid mechanics since eq.(1.6) is violated for a non-conducting fluid ($\eta \rightarrow \infty$) except for the case w = 0.

For the decelerating impulsive wall motion, it is observed that, in the case of equal diffusion coefficients, the shear stress at the solid-fluid interface and the disturbances in both the velocity and magnetic field increase with the increase of the wall conductivity or the applied magnetic field and with the decrease of the value of the parameter w. The speed of the diffusing Alfvén wave which transports the vorticity out of the boundary layer is modified to $(a^2 - 4\eta w)^{\frac{1}{2}}$, and the wave disturbance vanishes at $w \geq a^2/4\eta$, leaving the solution dominated by the effect of the viscous boundary layer continuously diffusing from the wall. The induced magnetic field is, in general, not negligible except for vanishingly small viscosity and zero wall conductivity, or for large values of the parameter w. An exact solution for the perfectly conducting, inviscid, incompressible fluid, obtained as the limiting case of equal diffusion coefficients with no restriction on the parameter w, shows that the Alfvén wave is emitted continuously from the solid-fluid interface. Boundary conditions are also discussed and the conditions for the approximation of a perfectly conducting wall or a non-conducting wall are given.

3. Results for accelerating wall motion

The impulsive start of a flat wall, considered as a limiting case of a very rapid acceleration, requires infinite starting stress at the solid-fluid interface and is primarily of mathematical interest. However, the problem of the gradually accelerating motion of a conducting wall starting from rest is of more physical interest since the shear stress at the solid-fluid interface is bounded. The solution follows directly from that of decelerating impulsive wall motion.

In the case of equal diffusion coefficients, the shear stress at the interface and the disturbances in both the velocity and magnetic field increase as time advances with the increase of the electrical conductivity of the wall or the values of the parameter w and with the decrease of the applied magnetic field. The disturbances in the velocity and magnetic field are small for small time. The exact solution for the case of a perfectly conducting, inviscid, incompressible fluid shows that the Alfvén wave is continuously emitted from the interface with zero intensity at the wave front.

4. General transient solution for a non-dissipative fluid

The solutions obtained for the decelerating impulsive motion of a perfectly conducting or non-conducting wall in the non-dissipative limit are only for the case of equal diffusion coefficients. The concept of the characteristic signal lines, as used by Leibovich and Ludford [15] and Ludford and Yannitell [16] in the thin airfoil problems, together with the exact solutions previously obtained, enables us to procede more generally. The transient solution for a non-dissipative fluid is obtained for an arbitrary wall motion with no restriction on the ratio of diffusivities. The characteristic approach used is discussed in Chapter V, and the solution for an oscillatory wall motion is given as an example.

II. BASIC EQUATIONS AND BOUNDARY CONDITIONS

A. Basic Equations

1. Dimensional governing equations

The extended problems considered here are governed by Maxwell's and Navier-Stokes' equations, together with electromagnetic as well as fluid dynamic boundary and initial conditions. All electromagnetic quantities will be referred to the fixed-in-space x-y-z coordinate system in the following analyses.

The equations governing the motion of the fluid in the absence of a pressure gradient, with a fixed-in-space x-y-z reference system, are given by Ludford [4] as

$$\left(\eta\frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial t}\right) B + B_{\circ}\frac{\partial U}{\partial y} = 0 , \tag{2.1}$$

$$a^2\frac{\partial B}{\partial y} + B_{\circ}\left(\nu\frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial t}\right) U = 0 . \tag{2.2}$$

The governing equation in the solid is

$$\frac{\partial B}{\partial t} = \eta_s \frac{\partial^2 B}{\partial y^2} . \tag{2.3}$$

Here all quantities are assumed to be functions of y and t. The motion is in the x-direction. The velocity $\underline{V}$, magnetic field intensity $\underline{B}$, and current $\underline{J}$ are given by

$$\underline{V} = (U,0,0),\ \underline{B} = (B,B_{\circ},0) \text{ and } \underline{J} = (0,0,j).$$

$B_{\circ}$ is a constant magnetic intensity perpendicular to the plane of motion. a is the Alfvén velocity defined as

$$a = B_\circ/(\rho\mu)^{\frac{1}{2}}$$

where ρ is the density and μ the magnetic permeability of the fluid. ν is the kinematic viscosity of the fluid. η and η_s are the magnetic diffusivity of the fluid and solid respectively and are defined as

$$\eta = \frac{1}{\mu\sigma}\ , \qquad \eta_s = \frac{1}{\mu_s\sigma_s}\ .$$

Here σ is the conductivity of the fluid, μ_s is the permeability of the solid. RMKS units are used throughout. The usual magnetohydrodynamics assumptions are used, i.e., the fluid velocity is assumed small compared with the velocity of light and hence the displacement currents are neglected, the fluid is assumed to be an incompressible, macroscopically neutral continuum with constant transport properties, and the conventional form of Ohm's law is employed.

Eq.(2.2) represents the combination of the balance of momentum and Ohm's law; eqs.(2.1) and (2.3) are from the equation of magnetic flux transport which is the combination of Faraday's law and Ampere's law, together with Ohm's law, with the electric field suppressed in favor of velocity and magnetic intensity.

The current j is given by Ampere's law,

$$j = -\frac{\partial B}{\partial y} \qquad (2.4)$$

2. Nondimensionalized governing equations

For convenience, nondimensional variables introduced by Steketee [11] will be used. We define the following quantities:

$$\xi = ay/\alpha, \quad \tau = a^2t/\alpha, \quad \gamma = (\beta/\alpha)^{\frac{1}{2}}, \quad u = U/a,$$

$$b = B/B_o, \quad \Omega = \alpha w/a^2 \quad \text{and} \quad \epsilon = (\nu/\eta)^{\frac{1}{2}},$$

where $2\alpha^{\frac{1}{2}} = \eta^{\frac{1}{2}} + \nu^{\frac{1}{2}}$ and $2\beta^{\frac{1}{2}} = \eta^{\frac{1}{2}} - \nu^{\frac{1}{2}}$. ϵ is related to the magnetic Prandtl number by

$$\epsilon = (\mu/\rho)^{\frac{1}{2}} \, Pr_m \, . \tag{2.5}$$

The reason for using the diffusion coefficient α, whose square root is the mean of the square roots of ν and η, to nondimensionalize the variables, instead of using a single diffusion coefficient ν or η, is that, by doing so, we are free to consider the limiting case of ν or η without having to redefine the nondimensional variables.

The introduction of the nondimensional variables into eqs.(2.1)-(2.3) leads to the set of equations;

$$\left\{(1 + \gamma)^2\frac{\partial^2}{\partial\xi^2} - \frac{\partial}{\partial\tau}\right\} b + \frac{\partial u}{\partial\xi} = 0 \, , \tag{2.6}$$

$$\left\{(1 - \gamma)^2\frac{\partial^2}{\partial\xi^2} - \frac{\partial}{\partial\tau}\right\} u + \frac{\partial b}{\partial\xi} = 0 \, , \tag{2.7}$$

$$\frac{\partial^2 b}{\partial\xi^2} = (\alpha/\eta_s) \, \frac{\partial b}{\partial\tau} \, . \tag{2.8}$$

Note that the linearity of the governing equations (2.6)-(2.8) suggest that superposition may be used to obtained various solutions for different boundary and initial value problems.

B. Boundary and Initial Conditions

1. Basic boundary and initial conditions

Across the solid-fluid interface, the velocity field and the tangential components of electric field must be continuous. Analytically, in non-dimensional form, this implies

$$(u)_{\xi = 0_+} = (u)_{\xi = 0_-} , \tag{2.9}$$

$$\left[(1 + \gamma)^2 \frac{\partial b}{\partial \xi}\right]_{\xi = 0_+} = \left[(\eta_s/\alpha)\, \frac{\partial b}{\partial \xi}\right]_{\xi = 0_-} . \tag{2.10}$$

If η and η_s are both non-zero (finite conductivity), the magnetic field must be continuous, i.e.,

$$(b)_{\xi = 0_+} = (b)_{\xi = 0_-} . \tag{2.11}$$

It is also assumed that, in transient problems, conditions at infinite distance from the solid-fluid interface will not be affected in finite time. Hence the velocity and magnetic intensity remain unchanged from their initial values at $y = \pm\infty$. This completes the set of boundary conditions for our problem.

The initial conditions on the velocity and magnetic intensity in the fluid will be

$$u = b = 0 \qquad \text{for all } \xi > 0. \tag{2.12}$$

2. Boundary conditions in limiting case

Laplace transform techniques are used throughout. The Laplace transform of the velocity, $\bar{u}$, and magnetic intensity, $\bar{b}$, are defined as

$$\bar{u}(\zeta,s) = \int_0^\infty e^{-st}u(\zeta,\tau)\ d\tau\ ,$$

$$\bar{b}(\zeta,s) = \int_0^\infty e^{-st}b(\zeta,\tau)\ d\tau$$

where s is the transform variable.

The general solution of transformed equations (2.6)-(2.8) satisfying the initial conditions, eq.(2.12), and the boundary conditions at infinity can be written as

$$\bar{b}(\zeta,s) = C_1\exp(-m\zeta) + C_2\exp(-n\zeta), \tag{2.13}$$

$$\bar{u}(\zeta,s) = (C_1/m)\left[(1+\gamma)^2m^2 - s\right]\exp(-m\zeta) + (C_2/n)\left[(1+\gamma)^2n^2 - s\right]\exp(-n\zeta) \tag{2.14}$$

in the fluid and

$$\bar{b}(\zeta,s) = C_3\exp\left[(\alpha s/\eta_s)^{\frac{1}{2}}\zeta\right] \tag{2.15}$$

in the solid, where

$$\begin{aligned} m &= \left[(1+4s)^{\frac{1}{2}} + (1+4\gamma^2 s)^{\frac{1}{2}}\right]/2(1-\gamma)^2, \\ n &= \left[(1+4s)^{\frac{1}{2}} - (1+4\gamma^2 s)^{\frac{1}{2}}\right]/2(1-\gamma)^2. \end{aligned} \tag{2.16}$$

The quantities C_1, C_2 and C_3 are functions of the transform variable s, and are determined by the boundary conditions at $\zeta = 0_+$.

Two limiting cases which greatly simplify our problem are considered here:

(i) Non-conducting wall ($\eta_s = \infty$) --- Physically this implies that the wall is a much better insulator than the fluid. Eq.(2.15) with $\eta_s = \infty$ reads

$$\bar{b} = C_3 , \tag{2.17}$$

which is independent of ζ. Since the magnetic field in the wall is assumed to be undisturbed at infinite distance from the interface in finite time, eq.(2.17) then implies

$$\bar{b} = 0 \qquad \text{for all } \zeta < 0,$$

or $$b = 0 \qquad \text{for all } \zeta < 0.$$

Eq.(2.1) then gives

$$(b)_{\zeta = 0_+} = 0 \tag{2.18}$$

which will be used as the boundary condition at the interface in the case of a non-conducting wall. These results may be applied in the cases,

$$(\eta_s/\alpha)^{\frac{1}{2}} >> 1,$$

or $$\eta_s/\eta >> \tfrac{1}{4}(1 + \epsilon)^2 \tag{2.19}$$

as the condition for approximating the non-conducting wall.

(ii) Perfectly conducting wall ($\eta_s = 0$) --- Physically this implies that the solid is a much better conductor than the fluid. Eq.(2.15) with $\eta_s = 0$ yields

$$\bar{b} = 0 \qquad \text{for all } \zeta < 0,$$

or $$b = 0 \qquad \text{for all } \zeta < 0. \tag{2.20}$$

Eq.(2.10) with $\eta_s = 0$ yields

$$\left(\frac{\partial b}{\partial \xi}\right)_{\xi = 0_+} = 0 \qquad (2.21)$$

which will be used as the boundary condition at the interface in the case of a perfectly conducting wall. These results may be applied in the cases,

$$(\eta_s/\alpha)^{\frac{1}{2}} << 1$$

or

$$\eta_s/\eta << \tfrac{1}{4}(1 + \epsilon)^2 \qquad (2.22)$$

as the condition for approximating the perfectly conducting wall. Eq.(2.22) with $\epsilon = 1$ becomes

$$\eta_s/\eta << 1$$

which is the condition used by Axford [8] in his treatment of the oscillating wall problem.

In these two limiting cases, the problem is simplified, and the transforms can be inverted in certain cases. The induced magnetic field in the wall is zero in both cases as in the MHD Rayleigh problem. The problem remaining is to solve the governing equations (2.6)-(2.7) together with the boundary condition at the interface, eq.(2.18) or eq.(2.21), in these two limiting cases.

III. DECELERATING IMPULSIVE WALL MOTION

A. Non-conducting Wall

1. General transformed solution

The problem of decelerating impulsive motion of a conducting flat wall may be approximated by that of a non-conducting wall in case the following condition is satisfied:

$$\eta_s/\eta \gg \tfrac{1}{4}(1 + \epsilon)^2 \; . \qquad (2.19)$$

The wall's velocity is assumed to be

$$u_{wall} = I(\tau)\, u_* \exp(-\Omega\tau) \; . \qquad (1.4)$$

The boundary conditions at the interface for a non-conducting wall are given by

$$(u)_{\xi = 0_+} = (u)_{\xi = 0_-} \; , \qquad (2.9)$$

$$(b)_{\xi = 0_+} = 0 \; . \qquad (2.18)$$

Eq.(2.9) together with eq.(1.4) yields

$$(u)_{\xi = 0_+} = I(\tau)\, u_* \exp(-\Omega\tau) \; . \qquad (3.1)$$

Transformed, eqs.(3.1) and (2.18) become

$$(\bar{u})_{\xi = 0_+} = u_*/(s + \Omega) \; , \qquad (3.2)$$

$$(\bar{b})_{\xi = 0_+} = 0 \; . \qquad (3.3)$$

The magnetic intensity, b, in the solid is given by

$$b = 0 \quad \text{in the solid} \qquad (2.17)$$

Eqs.(2.13) and (2.14), together with the boundary conditions, eqs.(3.2) and (3.3), give

$$C_1 = -C_2 = \frac{u_* mn}{(s + \Omega)(m - n)[s + (1 + \gamma)^2 mn]} \; .$$

Eqs.(2.13)-(2.14) then become

$$\bar{u}(\xi, s) = \frac{u_*}{s + \Omega}\left\{\frac{n[(1 + \gamma)^2 m^2 - s]}{(m - n)[(1 + \gamma)^2 mn + s]} \exp(-m\xi) - \frac{m[(1 + \gamma)^2 n^2 - s]}{(m - n)[(1 + \gamma)^2 mn + s]} \exp(-n\xi)\right\} , \quad (3.4)$$

$$\bar{b}(\xi, s) = \frac{u_* mn}{(s + \Omega)(m - n)[(1 + \gamma)^2 mn + s]}\left\{\exp(-m\xi) - \exp(-n\xi)\right\} \quad (3.5)$$

which are the general solutions of the transformed equations for the non-conducting wall.

2. Solutions in some special cases

The general solution of eqs.(3.4)-(3.5) for arbitrary values of ν, η and Ω is very complicated and can not be expressed in terms of tabulated functions. However, it is of interest to consider some special cases:

(1) $\nu = \eta \neq 0$

In the case with non-zero, equal diffusion coefficients we have $\gamma = 0$, and eqs.(2.16) become

$$m = (s + \tfrac{1}{4})^{\frac{1}{2}} + \tfrac{1}{2}$$

$$n = (s + \tfrac{1}{4})^{\frac{1}{2}} - \tfrac{1}{2}$$

Substitution in eqs.(3.4)-(3.5) yields

$$\begin{Bmatrix} \bar{u}(\xi,s) \\ \bar{b}(\xi,s) \end{Bmatrix} = \frac{u_*}{2(s+\Omega)} \left\{ \exp(-m\xi) \pm \exp(-n\xi) \right\} ,$$

or

$$\begin{Bmatrix} \bar{u}(\xi,s) \\ \bar{b}(\xi,s) \end{Bmatrix} = \frac{u_*}{s+\Omega} \begin{Bmatrix} \cosh(\xi/2) \\ -\sinh(\xi/2) \end{Bmatrix} \exp[-(s+\tfrac{1}{4})^{\frac{1}{2}}\xi] . \tag{3.6}$$

The exact solution is obtained for $\Omega \leq \frac{1}{4}$, by applying complex-Laplace-inversion to eqs.(3.6), in the appendix. The result is

$$\begin{Bmatrix} u(\xi,\tau) \\ b(\xi,\tau) \end{Bmatrix} = \tfrac{1}{2}u_* \exp(-\Omega\tau) \begin{Bmatrix} \cosh(\xi/2) \\ -\sinh(\xi/2) \end{Bmatrix} \Big\{ \exp[(1-4\Omega)^{\frac{1}{2}}\tfrac{\xi}{2}] \operatorname{erfc}\Big(\frac{\xi + (1-4\Omega)^{\frac{1}{2}}\tau}{2\tau^{\frac{1}{2}}}\Big) + \exp[-(1-4\Omega)^{\frac{1}{2}}\tfrac{\xi}{2}] \operatorname{erfc}\Big(\frac{\xi - (1-4\Omega)^{\frac{1}{2}}\tau}{2\tau^{\frac{1}{2}}}\Big) \Big\} \tag{3.7}$$

or

$$\begin{Bmatrix} U(y,t) \\ (\rho\mu)^{-\frac{1}{2}}B(y,t) \end{Bmatrix} = \tfrac{1}{2}U_* \exp(-wt) \begin{Bmatrix} \cosh(ay/2\eta) \\ -\sinh(ay/2\eta) \end{Bmatrix} \Big\{ \exp[(a^2-4\eta w)^{\frac{1}{2}}\tfrac{y}{2\eta}] \operatorname{erfc}\Big(\frac{y + (a^2-4\eta w)^{\frac{1}{2}}t}{2(\eta t)^{\frac{1}{2}}}\Big) + \exp[-(a^2-4\eta w)^{\frac{1}{2}}\tfrac{y}{2\eta}] \operatorname{erfc}\Big(\frac{y - (a^2-4\eta w)^{\frac{1}{2}}t}{2(\eta t)^{\frac{1}{2}}}\Big) \Big\} \tag{3.8}$$

in terms of dimensional variables. Three limiting cases are observed:

(a) $\Omega = w = 0$

The problem is reduced to the MHD Rayleigh problem with a non-conducting wall. Eqs.(3.8) with $w = 0$ yield

$$\begin{Bmatrix} U(y,t) \\ (\rho\mu)^{-\frac{1}{2}}B(y,t) \end{Bmatrix} = \tfrac{1}{2}U_{\circ}\begin{Bmatrix} \cosh(ay/2\eta) \\ -\sinh(ay/2\eta) \end{Bmatrix}\Big\{\exp(ay/2\eta)\,\mathrm{erfc}\Big(\frac{y+at}{2(\eta t)^{\frac{1}{2}}}\Big) + \exp(-ay/2\eta)\,\mathrm{erfc}\Big(\frac{y-at}{2(\eta t)^{\frac{1}{2}}}\Big)\Big\} . \tag{3.9}$$

This solution is in agreement with that given by Bryson and Rosciszewski [6].

(b) $w = a^2/4\eta$

Eqs.(3.8) with $w = a^2/4\eta$ yield

$$\begin{Bmatrix} U(y,t) \\ (\rho\mu)^{-\frac{1}{2}}B(y,t) \end{Bmatrix} = U_{\circ}\, e^{-wt}\mathrm{erfc}\Big(\frac{y}{2(\eta t)^{\frac{1}{2}}}\Big)\begin{Bmatrix} \cosh(ay/2\eta) \\ -\sinh(ay/2\eta) \end{Bmatrix} . \tag{3.10}$$

(c) $w = B_{\circ} = 0$

In this case, our problem is reduced to the classical Rayleigh problem. Eqs.(3.10) with $w = a = 0$ yield

$$U(y,t) = U_{\circ}\,\mathrm{erfc}\Big(\frac{y}{2(\nu t)^{\frac{1}{2}}}\Big) ,$$

$$B(y,t) = 0 .$$

This checks with the solution of the classical Rayleigh problem.

Eqs.(3.8)-(3.10) are plotted in fig.(4). It is ob-

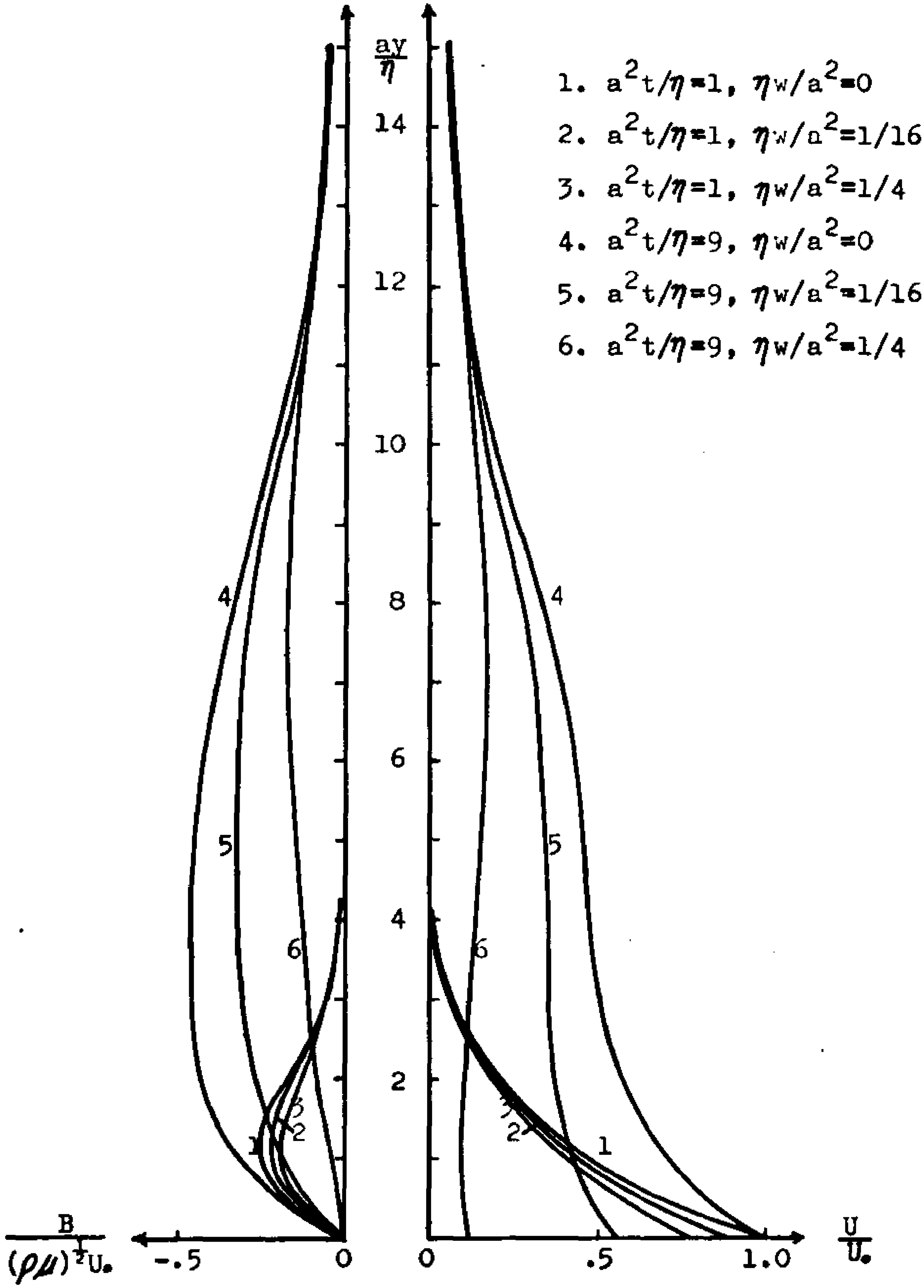

Fig. 4. Velocity and magnetic field profiles: decelerating impulsive motion of a non-conducting wall, $\nu = \eta \neq 0$.

served that the speed of the diffusing Alfvén wave, moving into the fluid, is modified to $(a^2 - 4\eta w)^{\frac{1}{2}}$. Also, as the value of the parameter Ω increases, the disturbances in both the velocity and magnetic field decrease; the wave disturbance vanishes at $\Omega \geqslant \frac{1}{4}$, leaving the flow dominated by the effect of the boundary layer diffusing from the wall. As the wave moves out of the viscous boundary layer, the flow behind the wave is, however, unsteady since the motion of the wall is not uniform, except for the case with $w = 0$ which is the MHD Rayleigh problem. The disturbance in the magnetic field may be assumed to be small for the case with $\Omega \geqslant \frac{1}{4}$, but is not negligible with $\Omega < \frac{1}{4}$ as shown in fig. (4).

(2) $\nu = \eta = 0$

This is the case of an incompressible, inviscid fluid with infinite conductivity and the parameter ϵ of unity. In the limiting process with $\nu = \eta \to 0$ in eq.(3.8), it is easy to show that

$$\lim_{\eta\to 0} \exp\left\{\left[a + (a^2 - 4\eta w)^{\frac{1}{2}}\right]\frac{y}{2\eta}\right\}\operatorname{erfc}\left(\frac{y + (a^2 - 4\eta w)^{\frac{1}{2}}t}{2(\eta t)^{\frac{1}{2}}}\right) = 0$$

by l'Hospital's rule and

$$\lim_{\eta\to 0} \exp\left\{\left[a - (a^2 - 4\eta w)^{\frac{1}{2}}\right]\frac{y}{2\eta}\right\}\operatorname{erfc}\left(\frac{y - (a^2 - 4\eta w)^{\frac{1}{2}}t}{2(\eta t)^{\frac{1}{2}}}\right)$$

$$= \exp(wy/a) \lim_{\eta\to 0} \operatorname{erfc}\left(\frac{y - (a^2 - 4\eta w)^{\frac{1}{2}}t}{2(\eta t)^{\frac{1}{2}}}\right)$$

$$= 2\ \exp(wy/a)\{1 - I(y-at)\}\ ,$$

$$\lim_{\eta\to 0} \exp\{-[a - (a^2 - 4\eta w)^{\frac{1}{2}}]\frac{y}{2\eta}\}\ \mathrm{erfc}\left(\frac{y + (a^2 - 4\eta w)^{\frac{1}{2}} t}{2(\eta t)^{\frac{1}{2}}}\right)$$
$$= 2\ \exp(-ay/a)\{1 - I(y-at)\}$$
$$= 0 \quad \text{for all } y > 0$$

by the binomial expansion. Eqs.(3.8) with $\nu = \eta \to 0$ then become

$$\left\{ \begin{matrix} U(y,t) \\ -(\rho\mu)^{-\frac{1}{2}}B(y,t) \end{matrix} \right\} = \tfrac{1}{2}U_0 \exp[w(y - at)/a]\{1 - I(y-at)\} \tag{3.11}$$

which is plotted in fig.(5).

This solution shows that the Alfvén wave travels in the positive y direction. The velocity distribution and induced magnetic field distribution in the wave region depend on the value of the Alfvén velocity a and the parameter w. Outside the wave region, the conditions are unchanged. Just behind the wave front, both (U/U_0) and $(-B/\sqrt{\rho\mu}U_0)$ are equal to $\frac{1}{2}$.

It is worthwhile noting that the Starwartson-Hartmann condition, eq.(1.3), which can be written as

$$[U - B/\epsilon(\rho\mu)^{\frac{1}{2}}]_{y\,=\,0_+} = U_0 \exp(-wt)$$

is satisfied in this special case. Also, the Alfvén condition, eq.(1.2), is satisfied across the wave front. The Alfvén wave is continuously emitted from the interface into the fluid because of the non-uniform motion of the wall. The strength of the velocity and induced magnetic field just

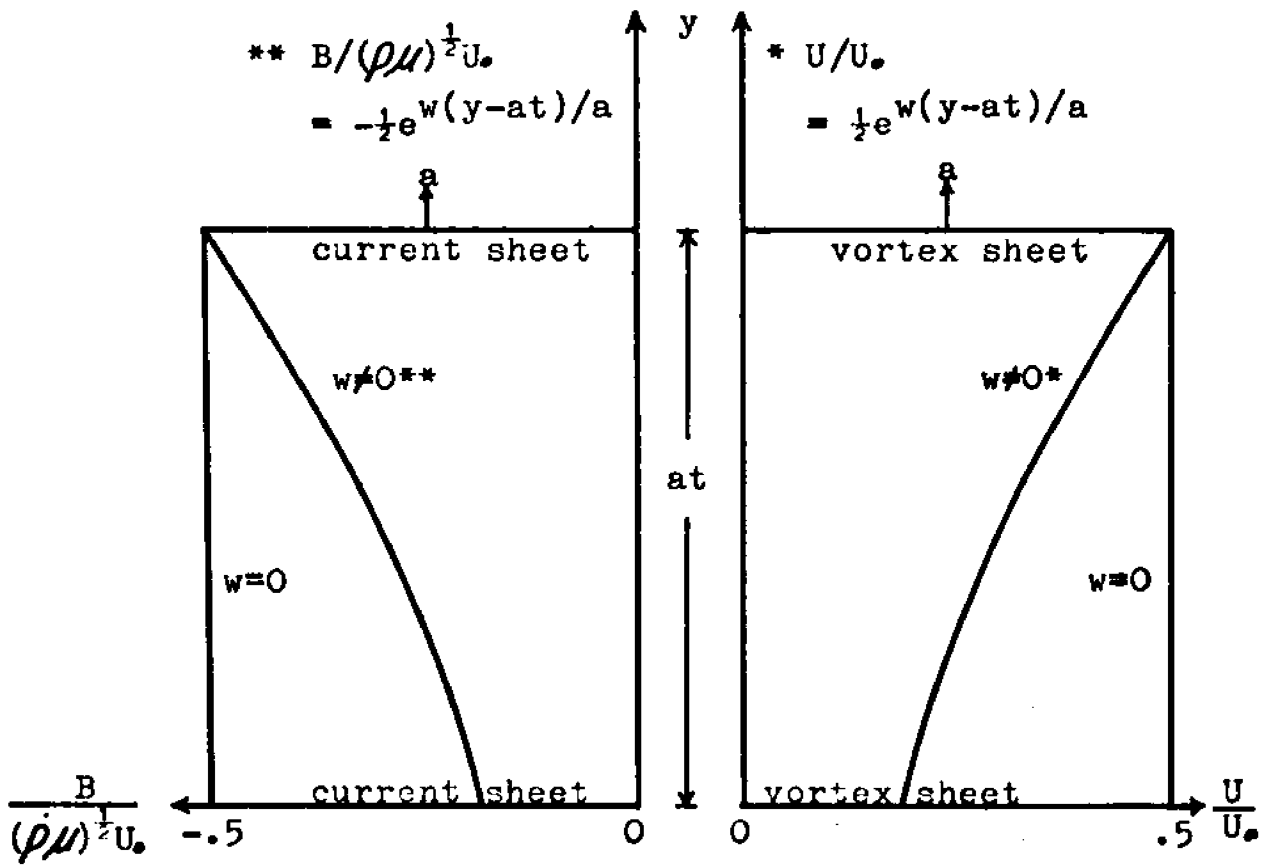

Fig. 5. Velocity and magnetic field profiles: decelerating impulsive motion of a non-conducting wall. $\nu = \eta = 0$.

behind the wave front is the solution of eqs.(1.2) and (1.3) at $t = 0_+$, $y = 0_+$ and remains unchanged.

The solution of eqs.(3.11) at y = 0 reads

$$\left\{ \begin{matrix} U \\ (\rho\mu)^{-\frac{1}{2}}B \end{matrix} \right\} = \pm \tfrac{1}{2}U_0 \exp(-wt)$$

indicating a slip at the interface with the strength of one half of the velocity of the wall and a jump in the tangential magnetic field.

As $w \to 0$, eqs.(3.11) become

$$\left\{ \begin{matrix} U(y,t) \\ (\rho\mu)^{-\frac{1}{2}}B(y,t) \end{matrix} \right\} = \pm \tfrac{1}{2}U_0 \left\{1 - I(y-at)\right\}$$

which checks with the solution given by Bryson and Rościszewski. Note that U = B = 0 as $B_0 = 0$ indicating complete slip at the interface; i.e., nothing happens in the fluid.

(3) $\epsilon << 1$

As this condition is satisfied by real fluids, this is the most practical case. In this case, γ may be approximated as

$$\gamma \doteq 1 - 2\epsilon \ .$$

Eqs.(2.13)-(2.14) then become

$$\bar{u}(\xi,s) = \frac{u_0}{s+\Omega}\left\{\exp[-(s+\tfrac{1}{4})^{\frac{1}{2}}\xi/2\epsilon] + \frac{\epsilon}{4(s+\tfrac{1}{4})} \exp(-\frac{s\xi}{2(s+\tfrac{1}{4})^{\frac{1}{2}}})\right\}, \qquad (3.12)$$

$$\bar{b}(\xi,s) \doteq \frac{\epsilon u_0}{2(s+\Omega)(s+\frac{1}{4})^{\frac{1}{2}}}\left\{\exp[-(s+\tfrac{1}{4})^{\frac{1}{2}}\xi/2\epsilon] - \exp\left(-\frac{s\xi}{2(s+\frac{1}{4})^{\frac{1}{2}}}\right)\right\}. \tag{3.13}$$

Note that as $\epsilon \to 0$, eqs.(3.12)-(3.13) imply that

$$u = b = 0$$

indicating complete slip at the interface and that there is no disturbance. For a vanishingly small value of the parameter ϵ, a thin viscous boundary layer forms at the interface.

The disturbance in magnetic field is negligible for vanishingly small ϵ and is independent of the applied field (at least to order ϵ); i.e.,

$$B = O(\epsilon)$$

from eq.(3.13). The solutions of eq.(3.12) are shown in the appendix to be

$$U(y,t) = \tfrac{1}{2}U_0 \exp(-wt)\left\{\exp[(a^2 - \eta w)^{\frac{1}{2}}y/\sqrt{\nu\eta}]\,\mathrm{erfc}[y/2\sqrt{\nu t} + \{(a^2-\eta w)t/\eta\}^{\frac{1}{2}}] + \exp[-(a^2-\eta w)^{\frac{1}{2}}y/\sqrt{\nu\eta}]\,\mathrm{erfc}[y/2\sqrt{\nu t} - \{(a^2-\eta w)t/\eta\}^{\frac{1}{2}}] + O(\epsilon)\right\} \tag{3.14}$$

for $w < a^2/\eta$,

$$U(y,t) = U_0\left\{\exp(-wt)\,\mathrm{erfc}(y/2\sqrt{\nu t}) + O(\epsilon)\right\} \tag{3.15}$$

for $w = a^2/\eta$, and

$$U(y,t) = \tfrac{1}{2}U_0\left\{\exp(ay/\sqrt{\nu\eta})\,\mathrm{erfc}(y/2\sqrt{\nu t} + a\sqrt{t/\eta}) + \exp(-ay/\sqrt{\nu\eta})\,\mathrm{erfc}(y/2\sqrt{\nu t} - a\sqrt{t/\eta}) + O(\epsilon)\right\}$$

(3.16)

for $w = 0$.

Eq.(3.16) is identical to Rossow's solution in which the induced magnetic field is assumed to be negligible in the case of the magnetic field fixed relative to the fluid. Note that with a value of the variable ζ greater than or equal to unity, eqs.(3.14)-(3.16) imply

$$u \doteq O(\epsilon) ,$$

indicating that the disturbance in the velocity field is limited to a thin viscous layer and is the superposition of two diffusing Alfvén waves moving in opposite directions. The wave disturbance in velocity in this layer decreases as the value of the parameter w increases. At $w = a^2/\eta$ ($\Omega = \frac{1}{4}$) the velocity distribution (U/U_{wall}) is identical to that of the classical Rayleigh problem (at least to order ϵ) in which the disturbance in the velocity field is due to the diffusion of the vorticity from the solid-fluid interface and is free of wave disturbance.

As $\nu \to 0$, eqs.(3.14)-(3.16) imply that

$$U/U_0 = O(\epsilon)$$

indicating complete slip at the interface as expected. As $\eta \to \infty$, eq.(3.16) yields

$$U(y,t) \doteq U_0 \operatorname{erfc}(y/2\sqrt{\nu t}) ,$$

which checks with eq.(1.1), the solution of the classical Rayleigh problem. The same result also can be obtained by setting $B_0 = 0$ in eq.(3.16).

3. Shear stress at the solid-fluid interface

The total shear force per unit area of the solid-fluid interface acting on the fluid is given by

$$F = -\rho\nu\left(\frac{\partial U}{\partial y}\right)_{y=0_+} - \frac{B_\bullet}{\mu}(B)_{y=0_+} \tag{3.17}$$

or

$$F = -\rho a^2(1-\gamma)^2\left(\frac{\partial u}{\partial \xi}\right)_{\xi=0_+} - \frac{B_\bullet^2}{\mu}(b)_{\xi=0_+} . \tag{3.18}$$

The first term on the right hand side of eq.(3.17) comes from skin friction and the second term is the magnetic force. For a non-conducting wall, no current is produced at the interface and hence the magnetic force vanishes; the shear stress at the interface comes only from skin friction.

Substituting eqs.(3.13) and (3.14) into eq.(3.18), together with eqs.(2.16), yields

$$\overline{C}_f = 2(1-\gamma)(s+\tfrac{1}{4})^{\frac{1}{2}}/u_\bullet(s+\Omega) \tag{3.19}$$

where $\overline{C}_f = \overline{F}/\frac{1}{2}\rho U_\bullet^2$ and $\overline{F}$ is the Laplace transform of F. The exact solutions of eq.(3.19) for $w \leq a^2/(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2$, i.e., $\Omega \leq \frac{1}{4}$, with arbitrary values of ν and η are

$$C_f = \frac{2}{U_\bullet}\Big\{(\nu/\pi t)^{\frac{1}{2}}\exp[-a^2t/(\nu^{\frac{1}{2}}+\eta^{\frac{1}{2}})^2] + \nu^{\frac{1}{2}}\exp(-wt)$$
$$\sqrt{a^2/(\nu^{\frac{1}{2}}+\eta^{\frac{1}{2}})^2 - w}\ \mathrm{erf}\left(\sqrt{(a^2/(\nu^{\frac{1}{2}}+\eta^{\frac{1}{2}})^2 - w)t}\right),$$
$$\text{for } w \leq a^2/(\nu^{\frac{1}{2}}+\eta^{\frac{1}{2}})^2, \tag{3.20}$$

$$C_f = \frac{2}{U_\bullet}(\nu/\pi t)^{\frac{1}{2}}\exp[-a^2t/(\nu^{\frac{1}{2}}+\eta^{\frac{1}{2}})^2]$$
$$\text{for } w = a^2/(\nu^{\frac{1}{2}}+\eta^{\frac{1}{2}})^2, \tag{3.21}$$

and

$$C_f = \frac{2}{U_*}(\nu/\pi t)^{\frac{1}{2}}\exp[-a^2t/(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2] + \frac{a\nu^{\frac{1}{2}}}{\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}}}$$

$$\mathrm{erf}(at^{\frac{1}{2}}/(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}}))$$

$$\text{for } w = 0, \qquad (3.22)$$

where $C_f = F/\frac{1}{2}\rho U_*^2$. Eqs.(3.20)-(3.22) are plotted in fig.(6) for $w(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2/4a^2 = 0$, 1/16, 1/8 and 1/4 with $\nu = \eta \neq 0$ (i.e., $\gamma = 0$). It is shown that the shear stress at the interface decreases as the value of the parameter $w(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2/4a^2$ increases. The limiting value of C_f is a/U_* for $w = 0$ and zero otherwise. Note that the stress at the interface vanishes for inviscid fluids since the only contribution is from the skin friction for a non-conducting wall.

4. Predicted approximate solution for $\Omega > \frac{1}{4}$

The general solution as mentioned previously is very complicated and only some special cases are considered. Even in these special cases, the solution, in general, is given only for

$$w \leq a^2/(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2 \qquad (3.23)$$

or $\Omega \leq \frac{1}{4}$. This can be expected since the closed-form solution in terms of known functions in the corresponding problem in ordinary fluid mechanics can not be found. It can be seen that the condition of eq.(3.23) is violated by setting $\eta = \infty$.

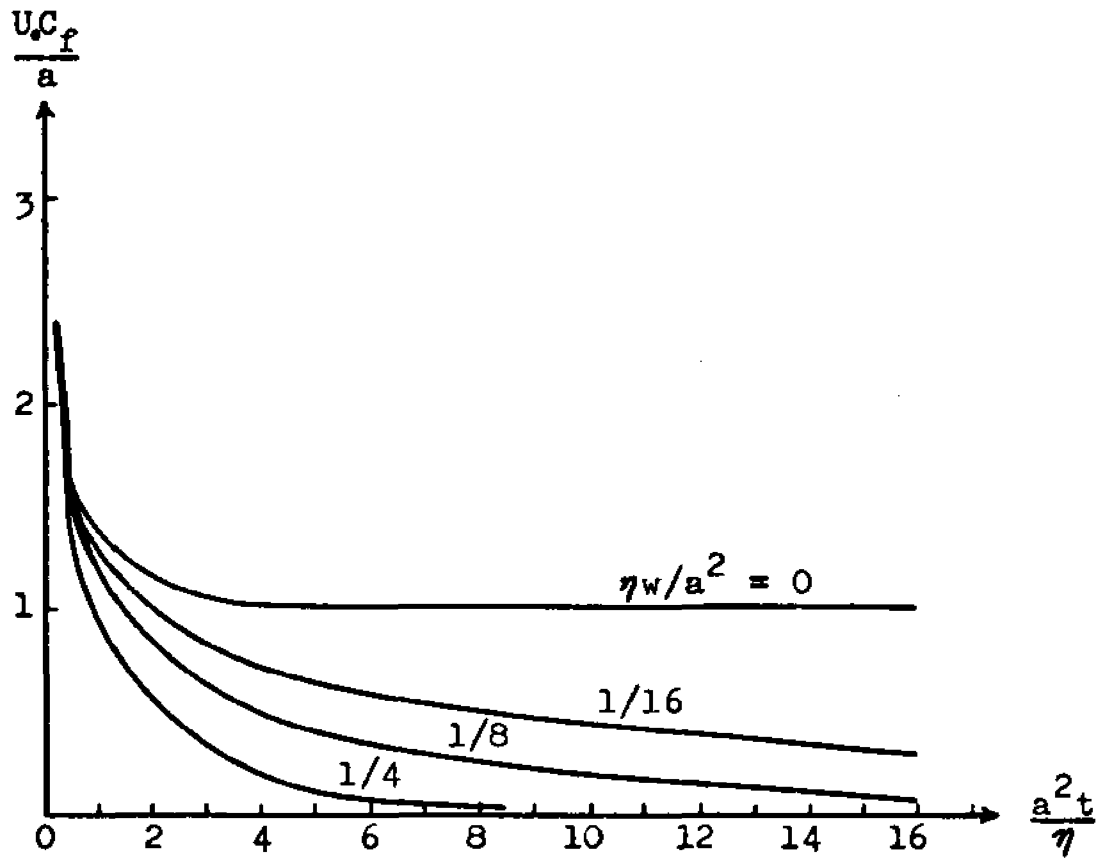

Fig. 6. Stress at the interface: decelerating impulsive motion of a non-conducting wall, $\nu = \eta \neq 0$.

Even though an exact solution is not available for $\Omega > \frac{1}{4}$, an approximate solution is suggested here. Since all curves of the velocity and magnetic field distribution, obtained for $\Omega \leq \frac{1}{4}$, are well-behaved and bounded between $\Omega = 0$ and $\Omega = \frac{1}{4}$, and since the solution is known for $\Omega = \infty$ (i.e., no motion), and since the flow for $\Omega \geq \frac{1}{4}$ is dominated by the viscous layer diffusing from the solid-fluid interface and is free of wave disturbances, it is predicted that the solutions of the velocity field and induced magnetic field can be approximated by eqs.(3.10) in the case of equal but non-zero diffusions, and eqs.(3.15) in the case of vanishingly small ϵ. For convenience, they are rewritten as follows:

(a) $\nu = \eta \neq 0$

$$\begin{Bmatrix} U(y,t) \\ (\rho\mu)^{-\frac{1}{2}}B(y,t) \end{Bmatrix} = U_0\, e^{-wt}\operatorname{erfc}(y/2\sqrt{\eta t})\begin{Bmatrix} \cosh(ay/2\eta) \\ \sinh(ay/2\eta) \end{Bmatrix}$$

$$\text{for } w \geq a^2/4\eta \; .$$

(b) $\epsilon \ll 1$

$$U(y,t) = U_0\left\{e^{-wt}\operatorname{erfc}(y/2\sqrt{\nu t}) + O(\epsilon)\right\} ,$$

$$B(y,t) = O(\epsilon)$$

$$\text{for } w \geq a^2/4\eta \; .$$

Note that there is no restriction on the value of w in the case of a perfectly conducting, inviscid, incompressible fluid in which eq.(3.23) is always satisfied for a non-zero applied magnetic field.

B. Perfectly Conducting Wall

1. General transformed solution

The problem of decelerating impulsive motion of a conducting flat wall may be approximated by that of a perfectly conducting wall if eq.(2.22) is satisfied, i.e.,

$$\eta_s/\eta \ll \tfrac{1}{4}(1 + \epsilon)^2 \ . \tag{2.22}$$

The boundary conditions at the interface for a perfectly conducting wall are given by

$$(u)_{\xi = 0_+} = I(\tau)u_* \exp(-\Omega\tau) \ ,$$

$$\left(\frac{\partial b}{\partial \xi}\right)_{\xi = 0_+} = 0,$$

since the wall velocity is assumed to be

$$u_{wall} = I(\tau)u_* \exp(-\Omega\tau) \ . \tag{1.4}$$

Transformed, they become

$$(\bar{u})_{\xi = 0_+} = u_*/(s + \Omega) \ , \tag{3.24}$$

$$\left(\frac{\partial \bar{b}}{\partial \xi}\right)_{\xi = 0_+} = 0 \ . \tag{3.25}$$

The induced magnetic intensity is given by

$$b = 0 \quad \text{in the solid} \tag{2.20}$$

Eqs.(2.13) and (2.14), together with the boundary conditions of eqs.(3.24) and (3.25), yield

$$C_1 = -(n/m)C_2 = \frac{u_* mn^2}{s(s + \Omega)(m^2 - n^2)} \ .$$

Substituting C_1 and C_2 into eqs.(2.13) and (2.14) yields

$$\bar{u}(\xi,s) = \frac{u_0}{s(s+\Omega)(m^2-n^2)}\Big\{-n^2[s-(1+\gamma)^2m^2]e^{-m\xi}$$

$$+ m^2[s^2-(1+\gamma)^2n^2]e^{-n\xi}\Big\}, \qquad (3.26)$$

$$\bar{b}(\xi,s) = \frac{u_0\, mn}{s(s+\Omega)(m^2-n^2)}\Big\{n\, e^{-m\xi} - m\, e^{-n\xi}\Big\}$$

$$(3.27)$$

which are the general transformed solutions for the perfectly conducting wall. m and n are given by eqs.(2.16).

2. Solutions in some special cases

The general solution is again very complicated and can not be expressed in terms of known functions. Some special cases are considered:

$$(1)\ \nu = \eta \neq 0$$

Eqs.(3.26) and (3.27), together with eqs.(2.16), for the case of equal but non-vanishing diffusion coefficients read

$$\begin{Bmatrix}\bar{u}(\xi,s)\\ \bar{b}(\xi,s)\end{Bmatrix} = \frac{u_0}{2(s+\Omega)}\Big[1-\tfrac{1}{2}(s+\tfrac{1}{4})^{-\frac{1}{2}}\Big]\exp\Big[\{-(s-\tfrac{1}{4})^{\frac{1}{2}}$$

$$-\tfrac{1}{2}\}\xi\Big] \pm \Big[1+\tfrac{1}{2}(s+\tfrac{1}{4})^{-\frac{1}{2}}\Big]\exp\Big[\{-(s+\tfrac{1}{4})^{\frac{1}{2}}$$

$$+\tfrac{1}{2}\}\xi\Big]\ . \qquad (3.28)$$

The exact solution with $\Omega < \frac{1}{4}$ is found by applying complex-Laplace-inversion to eqs.(3.28), (see appendix). The solution is

$$\left\{\begin{matrix} U(y,t) \\ (\rho\mu)^{-\frac{1}{2}}B(y,t) \end{matrix}\right\}$$

$$= \pm \tfrac{1}{2}U_0\, e^{-wt}\left\{\left\{\begin{matrix} \cosh(ay/2\eta) - a(a^2 - 4\eta w)^{-\frac{1}{2}}\sinh(ay/2\eta) \\ \sinh(ay/2\eta) - a(a^2 - 4\eta w)^{-\frac{1}{2}}\cosh(ay/2\eta) \end{matrix}\right\}\right.$$

$$\exp\left[(a^2 - 4\eta w)^{\frac{1}{2}}\frac{y}{2\eta}\right] \mathrm{erfc}\left(\frac{y + (a^2 - 4\eta w)^{\frac{1}{2}}t}{2(\eta t)^{\frac{1}{2}}}\right) +$$

$$\left\{\begin{matrix} \cosh(ay/2\eta) + a(a^2 - 4\eta w)^{-\frac{1}{2}}\sinh(ay/2\eta) \\ \sinh(ay/2\eta) + a(a^2 - 4\eta w)^{-\frac{1}{2}}\cosh(ay/2\eta) \end{matrix}\right\}$$

$$\left.\exp\left[-(a^2 - 4\eta w)^{\frac{1}{2}}\frac{y}{2\eta}\right] \mathrm{erfc}\left(\frac{y - (a^2 - 4\eta w)^{\frac{1}{2}}t}{2(\eta t)^{\frac{1}{2}}}\right)\right\} \tag{3.29}$$

in terms of dimensional variables. Three limiting cases are observed:

(a) $\Omega = w = 0$

This is the MHD Rayleigh problem with a perfectly conducting wall. Eqs.(3.29) with $w = 0$ yield

$$\left\{\begin{matrix} U(y,t) \\ (\rho\mu)^{-\frac{1}{2}}B(y,t) \end{matrix}\right\} = \tfrac{1}{2}U_0\left\{\mathrm{erfc}\left(\frac{y + at}{2(\eta t)^{\frac{1}{2}}}\right) \pm \mathrm{erfc}\left(\frac{y - at}{2(\eta t)^{\frac{1}{2}}}\right)\right\} . \tag{3.30}$$

This solution checks with that of several earlier investigators [3, 4, 6].

(b) $w = a^2/4\eta$ $(\Omega = \frac{1}{4})$

Eqs.(3.29) with $w = a^2/4\eta$ yield

$$\left\{\begin{matrix} U(y,t) \\ (\rho\mu)^{-\frac{1}{2}}B(y,t) \end{matrix}\right\}$$

$$= \pm U_0 e^{-wt} \begin{Bmatrix} \cosh(ay/2\eta) - (ay/2\eta)\sinh(ay/2\eta) \\ \sinh(ay/2\eta) - (ay/2\eta)\cosh(ay/2\eta) \end{Bmatrix}$$

$$\operatorname{erfc}(y/2\sqrt{\eta t}) + (at/\pi\eta)^{\frac{1}{2}} \begin{Bmatrix} \sinh(ay/2\eta) \\ \cosh(ay/2\eta) \end{Bmatrix} \exp(-y^2/4\eta t) \tag{3.31}$$

by l'Hospital's rule. This solution also can be obtained by applying the complex-Laplace-inversion to eqs.(3.26) and (3.27) with $\eta w/a^2 = \frac{1}{4}$.

(c) $w = B_0 = 0$

Again the problem is reduced to the classical Rayleigh problem. Eqs.(3.31) with $w = a = 0$ yield

$U(y,t) = U_0 \operatorname{erfc}(y/2\sqrt{\nu t})$,

$B(y,t) = 0$.

This checks with the known solution.

Eqs.(3.29)-(3.31) are plotted in fig.(7). It is observed that the situation is similar to that for a non-conducting wall. The speed of the diffusing Alfvén wave is modified to $(a^2 - 4\eta w)^{\frac{1}{2}}$. The wave disturbance vanishes if $\eta w/a^2 \geq \frac{1}{4}$ so that the flow is dominated by the effect of the viscous boundary layer diffusing from the interface. The disturbances in the velocity and magnetic field decrease with the increase of the value of the parameter $(\eta w/a^2)$. However, the velocity and induced magnetic field are larger than the corresponding fields for the non-conducting wall, and the disturbance in magnetic field is not negligible

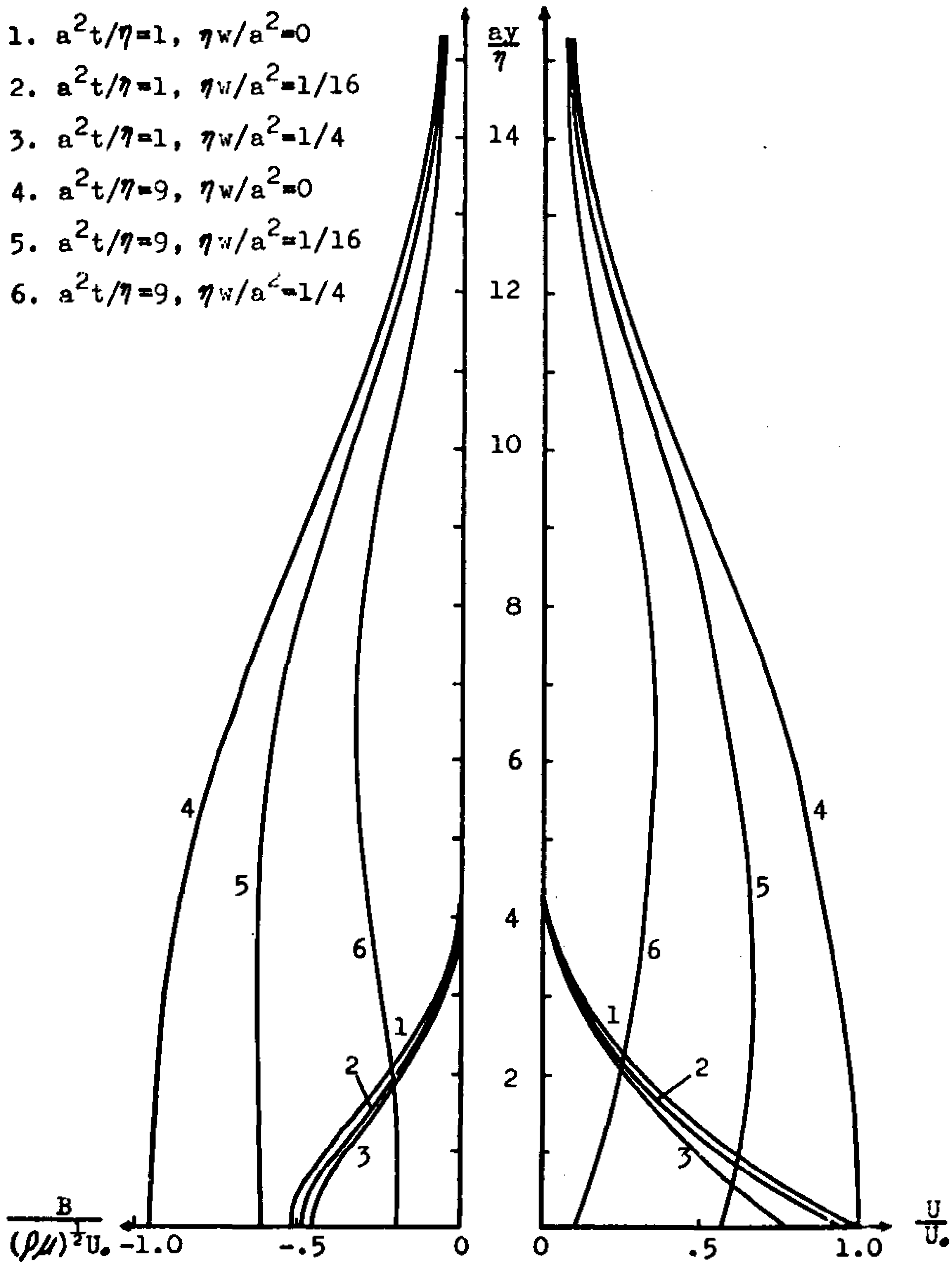

Fig. 7. Velocity and magnetic field profiles: decelerating impulsive motion of a perfectly conducting wall, $\nu = \eta \neq 0$.

except for $\eta w/a^2 >> \frac{1}{4}$.

$$(2)\ \nu = \eta = 0$$

This case is similar to that for a non-conducting wall. Eqs.(3.29) with $\nu = \eta \to 0$ read

$$\begin{Bmatrix} U(y,t) \\ (\rho\mu)^{-\frac{1}{2}}B(y,t) \end{Bmatrix} = \pm\, U_0 \exp[w(y - at)/a]\{1 - I(y-at)\} \tag{3.32}$$

which are plotted in fig.(8).

This solution shows that both the velocity and induced magnetic field have twice the value of the solution for the non-conducting wall. Just behind the wave front, both the values (U/U_0) and $(-B/\sqrt{\rho\mu}U_0)$ are equal to unity. The velocity at the interface obtained from eqs.(3.32) is

$$U(0,t) = U_0 \exp(-wt)$$

indicating that there is no slip at the interface. However, there is a jump in the tangential component of the magnetic field at the interface due to a current sheet in the wall;

$$B(0,t) = -(\rho\mu)^{\frac{1}{2}}U_0 \exp(-wt)\ .$$

The Alfvén condition, eq.(1.2), also is satisfied across the wave front as in the previous problem. However, the Stewartson-Hartmann condition, eq.(1.3), can not be applied in this case; the continuity of the tangential electric field is used to replace it. Since $\underline{E} = -\underline{V} \times \underline{B}$ for infinite conductivity, the only component of the electric field is in the z-direction;

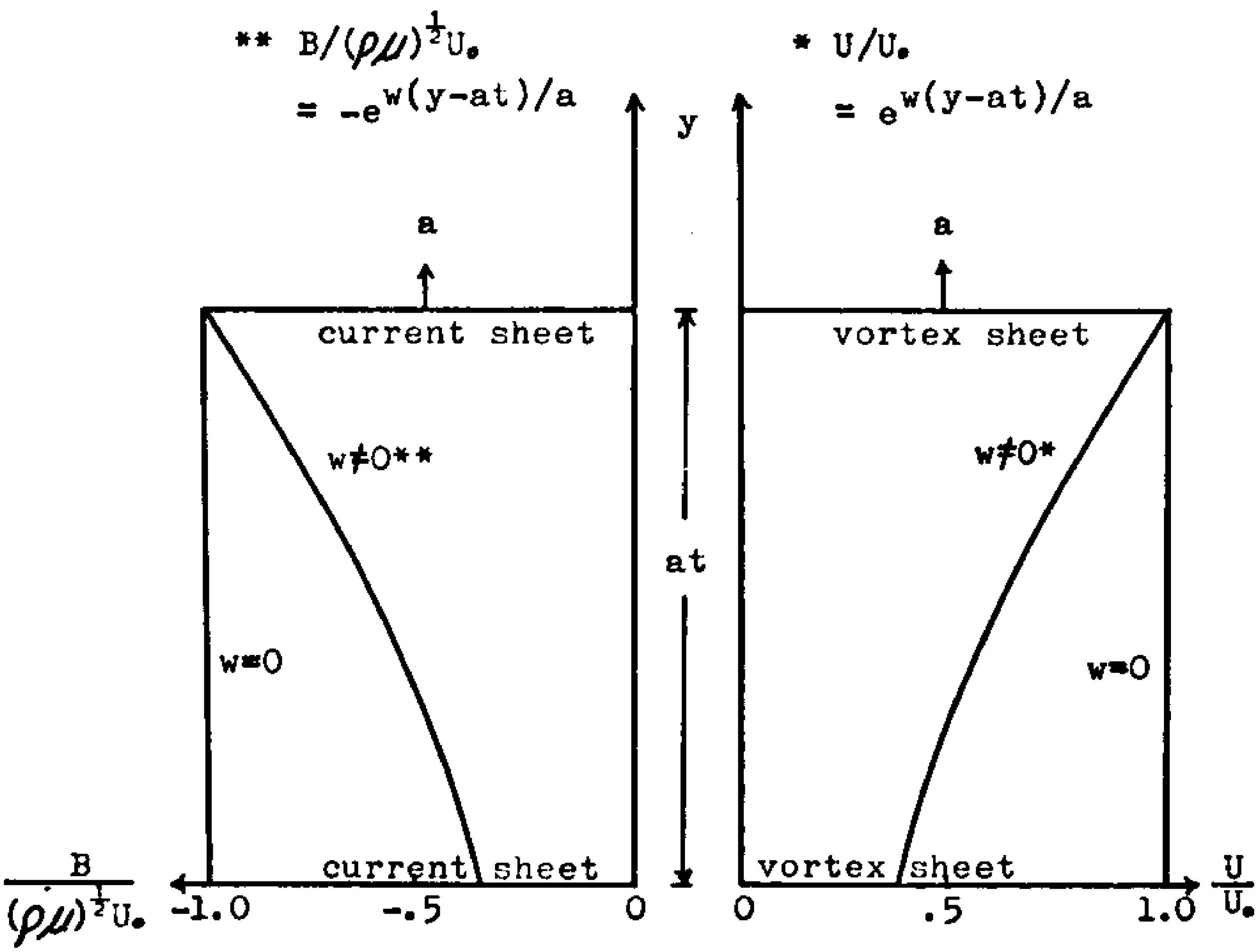

Fig. 8. Velocity and magnetic field profiles: decelerating impulsive motion of a perfectly conducting wall, $\nu = \eta = 0$.

$$E = -UB_0 \ .$$

Since the applied field is a constant, the continuity of the tangential electric field then reads

$$U(0,t) = U_0 \exp(-wt)$$

which checks with eq.(3.33).

Eqs.(3.32) with $w = 0$ read

$$\left\{ \begin{array}{c} U(y,t) \\ (\rho\mu)^{-\frac{1}{2}} B(y,t) \end{array} \right\} = \pm\, U_0 \left\{ 1 - I(y-at) \right\}$$

which checks with the solution of Bryson and Rościszewski.

(3) $\epsilon \ll 1$

Eqs.(3.26) and (3.27) with $\gamma \doteq 1 - 2\epsilon$ read

$$\bar{u}(\xi,s) = \frac{u_0}{s+\Omega} \left\{ \frac{s}{s+\frac{1}{4}} \exp[-(s+\tfrac{1}{4})^{\frac{1}{2}}\xi/2\epsilon] + \right.$$

$$\left. \frac{1}{4s+1} \exp[-s(s+\tfrac{1}{4})^{-\frac{1}{2}}\xi/2] \right\} , \qquad (3.34)$$

$$\bar{b}(\xi,s) = \frac{u_0}{s+\Omega} \left\{ \tfrac{1}{2}\epsilon s(s+\tfrac{1}{4})^{-3/2} \exp[-(s+\tfrac{1}{4})^{\frac{1}{2}}\xi/2\epsilon] - \right.$$

$$\left. \frac{1}{4s+1} \exp[-s(s-\tfrac{1}{4})^{-\frac{1}{2}}\xi/2] \right\} . \qquad (3.35)$$

as $\epsilon \to 0$, eq.(3.34) becomes

$$\bar{u}(\xi,s) = \frac{u_0}{(s+\Omega)(4s+1)} \exp[-s(s+\tfrac{1}{4})^{-\frac{1}{2}}\xi/2] \ .$$

The solution at the interface is

$$U(0,t) = \frac{U_0}{1 - \eta w/a^2} \left\{ \exp(-wt) - \exp(-a^2 t/\eta) \right\}$$

$$\text{for } w \neq a^2/\eta \ ,$$

$$U(0,t) = (U_0 a^2 t/\eta)\exp(-wt)$$
$$\text{for } w = a^2/\eta \;.$$

which shows that there is a slip at the interface with the value $S(w,t)$;

$$S(w,t) = \frac{U_0}{1 - \eta w/a^2}\left\{\exp(-a^2t/\eta) - (\eta w/a^2)\exp(-wt)\right\}$$
$$\text{for } w \neq a^2/\eta \;, \tag{3.36}$$

$$S(w,t) = U_0(1 - a^2t/\eta)\exp(-wt)$$
$$\text{for } w = a^2/\eta \;. \tag{3.37}$$

Eqs.(3.36)-(3.37) are plotted in fig.(9). It is observed that there is a complete slip at $t = 0$. As time advances, the slip decrease exponentially for $w = 0$ and decrease more rapidly, becoming negative, with an increase in the value of w. This indicates that a thin viscous layer exists near the interface for a vanishingly small value of ϵ, and this viscous layer is more important for small time.

Eqs.(3.34) and (3.35) can be approximated as

$$\bar{u}(\xi,s) \doteq \frac{u_0}{s+\Omega}\,\frac{s}{s+\frac{1}{4}}\exp[-(s+\tfrac{1}{4})^{\frac{1}{2}}\xi/2\epsilon] + \frac{1}{4s+1} \;, \tag{3.38}$$

$$\bar{b}(\xi,s) \doteq -\, u_0/(s+\Omega)(4s+1)^{\frac{1}{2}} \tag{3.39}$$

for small ξ. The approximate solutions of eqs.(3.38) and (3.39) are

$$U(y,t) \doteq U_0\exp(-wt) - a^2(a^2 - \eta w)^{-1}U_0\Big\{\exp(-a^2t/\eta)$$
$$\operatorname{erf}(y/2\sqrt{\nu t}) - \tfrac{1}{2}(\eta w/a^2)\exp(-wt)\Big[$$

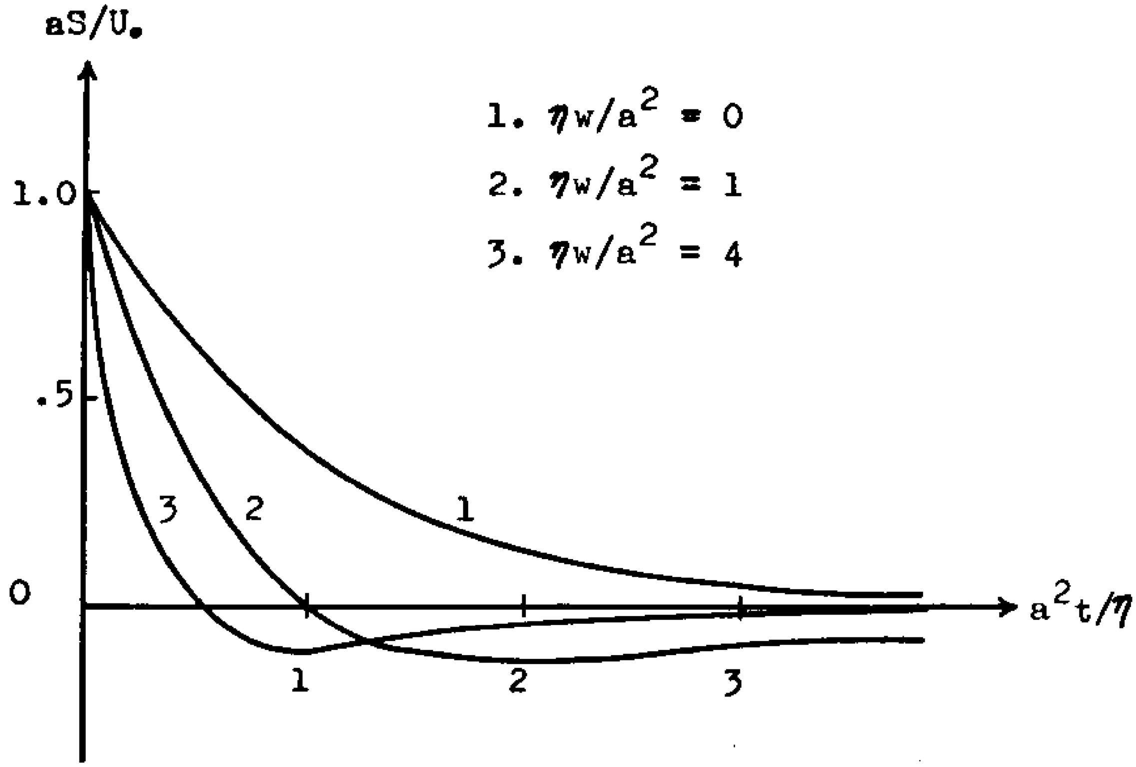

Fig. 9. Slip at the interface: decelerating impulsive motion of a perfectly conducting wall, $\epsilon = 0$.

$$\exp((a^2 - \eta w)^{\frac{1}{2}}y/\sqrt{\nu\eta})\ \mathrm{erfc}(y/2\sqrt{\nu t} + \{(a^2 - \eta w)t/\eta\}^{\frac{1}{2}}) + \exp(-(a^2 - \eta w)^{\frac{1}{2}}y/\sqrt{\nu\eta})\ \mathrm{erfc}(y/2\sqrt{\nu t} - \{(a^2 - \eta w)t/\eta\}^{\frac{1}{2}}) - 2]\Big\}$$

$$\text{for } w < a^2/\eta\ , \tag{3.40}$$

$$U(y,t) = U_* \exp(-wt) - U_* \exp(-a^2t/\eta)\Big\{(1 - a^2t/\eta)\ \mathrm{erf}(y/2\sqrt{\nu t}) + (a^2y^2/2\nu\eta)\mathrm{erfc}(y/2\sqrt{\nu t}) - (4a^2y/\eta)(t/\pi\nu)^{\frac{1}{2}}\exp(-y^2/4\nu t)\Big\}$$

$$\text{for } w = a^2/\eta\ ; \tag{3.41}$$

and

$$B(y,t) = -(\rho\mu)^{\frac{1}{2}}U_* a(a^2 - \eta w)^{-\frac{1}{2}}\exp(-wt)\ \mathrm{erf}(\{(a^2 - \eta w)t/\eta\}^{\frac{1}{2}})$$

$$\text{for } w < a^2/\eta\ , \tag{3.42}$$

$$B(y,t) = -2(\rho\mu)^{\frac{1}{2}}U_*(a^2t/\pi\eta)^{\frac{1}{2}}\exp(-a^2t/\eta)$$

$$\text{for } w = a^2/\eta\ . \tag{3.43}$$

Eqs.(3.40) and (3.42), with $w = 0$, read

$$U(y,t) = U_*\Big\{1 - \exp(-a^2t/\eta)\mathrm{erf}(y/2\sqrt{\nu t})\Big\}\ ,$$

$$B(y,t) = -(\rho\mu)^{\frac{1}{2}}U_* \mathrm{erf}(a\sqrt{t/\eta})$$

which check with the solution of Ludford. The method of steepest descent may be used to approximate the complex inversion integrals of eqs.(3.34) and (3.35) for $\xi \gg 1$. The disturbances in the velocity and magnetic field outside the viscous layer are not negligible and are greatly affected by the diffusing Alfvén wave.

3. Shear stress at the solid-fluid interface

The total shear force per unit area of the interface acting on the fluid is governed by eq.(3.17) or (3.18). Substitution of eqs.(3.26) and (3.27) into eq.(3.18), together with eqs.(2.16), yields

$$\bar{C}_f = 4(1-\gamma)s/u_*(s+\Omega)(s+\tfrac{1}{4})^{\frac{1}{2}} + 1/u_*(s+\Omega)(s+\tfrac{1}{4})^{\frac{1}{2}} \; . \qquad (3.44)$$

The first term on the right hand side of eq.(3.44) comes from skin friction and the second term from the magnetic force. The solution of eq.(3.44) are

$$C_f = \frac{2}{U_*}(\nu/\pi t)^{\frac{1}{2}}\exp\left[-a^2t/(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2\right. + \frac{2a^2}{U_*(a^2 - (\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2 w)^{\frac{1}{2}}}\left\{1 - \nu^{\frac{1}{2}}(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})w/a^2\right\} \exp(-wt)\ \mathrm{erf}\left(\sqrt{(a^2/(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2 - w)t}\right)$$

$$\text{for } w < a^2/(\nu^{\frac{1}{2}} - \eta^{\frac{1}{2}})^2 \; , \qquad (3.45)$$

$$C_f = \exp\left[-a^2t/(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2\right] \frac{2}{U_*}(\nu/\pi t)^{\frac{1}{2}} + 4a^2(\eta t/\pi)^{\frac{1}{2}}/U_*(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2$$

$$\text{for } w = a^2/(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2 \; , \qquad (3.46)$$

and

$$C_f = \frac{2}{U_*}(\nu/\pi t)^{\frac{1}{2}}\exp\left[-a^2t/(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2\right. +$$

$$(2a/U_\bullet)\,\mathrm{erf}(at^{\frac{1}{2}}/(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}}))$$

$$\text{for } w = 0 \; . \qquad (3.47)$$

Note that eq.(3.47) checks with the solution of Bryson and Rościszewski.

Eqs.(3.45)-(3.47) are plotted in fig.(10) for $\nu = \eta \neq 0$. It is shown that the shear stress at the interface decreases with the increase of the value $w(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2/4a^2$ as in the case of a non-conducting wall. However, the shear stress at a perfectly conducting wall is larger than the corresponding stress at a non-conducting wall. The limiting value of $(U_\bullet C_f/a)$ is two for $w = 0$ and zero otherwise as shown in fig.(10).

The shear stress at the interface with $\nu = \eta = 0$ is

$$C_f = a \exp(-wt)/U_\bullet$$

which comes from the magnetic force since the friction force vanishes for the inviscid fluid as shown in eq.(3.17).

4. Predicted approximate solution for $\Omega > \frac{1}{4}$

We see that the effect of the electrical conductivity of the wall greatly increases the disturbances in both the velocity and magnetic field. The induced magnetic field is not negligible even for vanishingly small value of ϵ. The solution with $\Omega > \frac{1}{4}$ again is not available. By the previous reasoning, it is predicted that the solution of the velocity and induced magnetic field can be approximated by eq.(3.31)

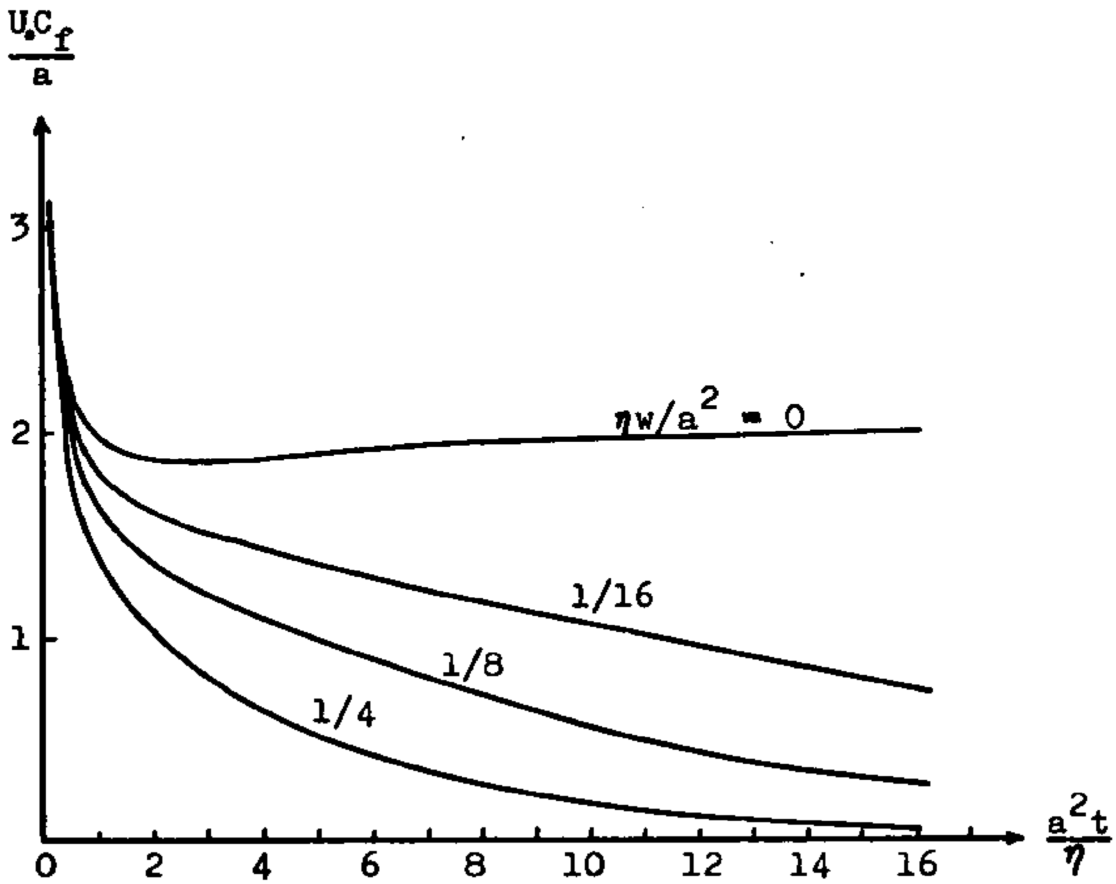

Fig. 10. Stress at the interface: decelerating impulsive motion of a perfectly conducting wall, $\nu = \eta = 0$.

in the case of equal but non-zero diffusion coefficients and eqs.(3.41) and (3.43) in the case of vanishingly small ϵ.

IV. ACCELERATING WALL MOTION

A. Non-conducting Wall

1. Boundary conditions at the solid-fluid interface

The boundary conditions at the interface for the accelerating motion of a non-conducting wall are given by eqs. (2.9) and (2.18), together with eq.(1.5), as

$$(u)_{\xi = 0_+} = I(\tau)u_*[1 - \exp(-\Omega\tau)] \quad , \tag{4.1}$$

$$(b)_{\xi = 0_+} = 0 \ . \tag{4.2}$$

Because of the linearity of the governing eqs.(2.1)-(2.3), the solution comes immediately from those of Chapt. III, A by the method of superposition. Some particular solutions are listed in the next section.

2. Solutions in some special cases

(1) $\nu = \eta \neq 0$

The solution for the case of equal but non-zero diffusion coefficients with $w \leq a^2/4\eta$ is

$$\begin{Bmatrix} U(y,t) \\ (\rho\mu)^{-\frac{1}{2}}B(y,t) \end{Bmatrix}$$

$$= \tfrac{1}{4}U_* \begin{Bmatrix} \cosh(ay/2\eta) \\ -\sinh(ay/2\eta) \end{Bmatrix} \Big\{ \exp(ay/2\eta)\operatorname{erfc}\Big(\frac{y + at}{2(\eta t)^{\frac{1}{2}}}\Big)$$

$$+ \exp(-ay/2\eta)\operatorname{erfc}\Big(\frac{y - at}{2(\eta t)^{\frac{1}{2}}}\Big) - \exp(-wt)\Big[$$

$$\exp((a^2 - 4\eta w)^{\frac{1}{2}}y/2\eta)\text{erfc}(\frac{y + (a^2 - 4\eta w)^{\frac{1}{2}}t}{2(\eta t)^{\frac{1}{2}}}) +$$

$$\exp(-(a^2 - 4\eta w)^{\frac{1}{2}}y/2\eta)\text{erfc}(\frac{y - (a^2 - 4\eta w)^{\frac{1}{2}}t}{2(\eta t)^{\frac{1}{2}}})$$

$$\text{for } w < a^2/4\eta \,, \tag{4.3}$$

and

$$\begin{Bmatrix} U(y,t) \\ (\rho\mu)^{-\frac{1}{2}}B(y,t) \end{Bmatrix}$$

$$= \tfrac{1}{2}U_0 \begin{Bmatrix} \cosh(ay/2\eta) \\ -\sinh(ay/2\eta) \end{Bmatrix}\Big\{\exp(ay/2\eta)\text{erfc}(\frac{y + at}{2(\eta t)^{\frac{1}{2}}}) +$$

$$\exp(-ay/2\eta)\text{erfc}(\frac{y - at}{2(\eta t)^{\frac{1}{2}}}) - 2\exp(-wt)\text{erfc}(y/2\sqrt{\eta t})\Big\}$$

$$\text{for } w = a^2/4\eta \,. \tag{4.4}$$

Eqs.(4.3)-(4.4) are plotted in fig.(11) for $a^2t/\eta = 1$ and 9 with $\eta w/a^2 = 1/4$ and 1/16. It is observed that the disturbances in both the velocity and magnetic field increase as time advances, with the corresponding solution of the MHD Rayleigh problem as its limit for time approaching infinity. The induced magnetic field is not negligible except at small time.

(2) $\nu = \eta = 0$

The solution for the case of an incompressible, inviscid, perfectly conducting fluid with $\epsilon = 1$ is

$$\begin{Bmatrix} U(y,t) \\ (\rho\mu)^{-\frac{1}{2}}B(y,t) \end{Bmatrix}$$

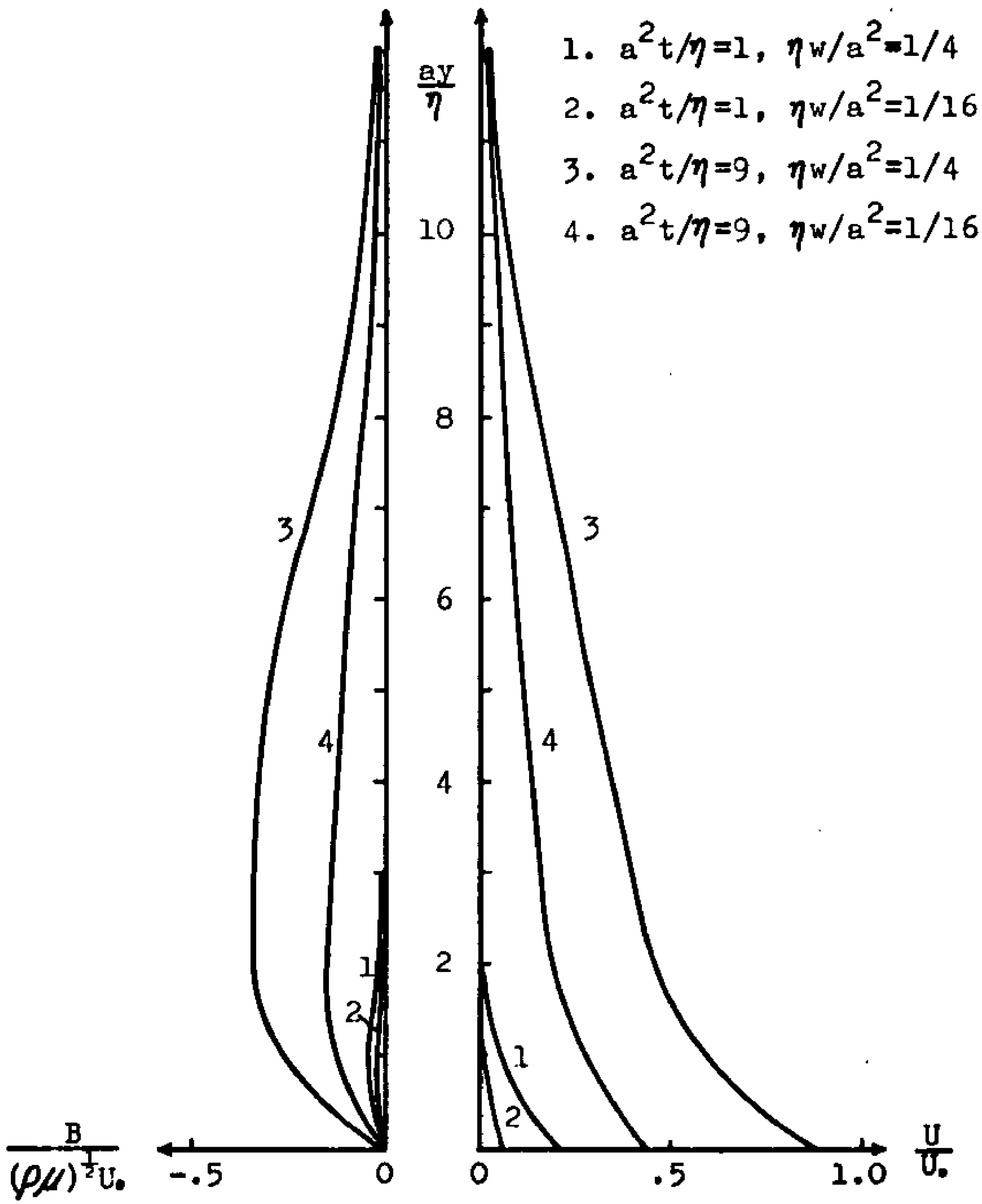

Fig. 11. Velocity and magnetic field profiles: accelerating motion of a non-conducting wall, $\nu = \eta \neq 0$.

$$= \pm \tfrac{1}{2}U_o \left\{1 - \exp(w(y - at)/a)\right\}\left\{1 - I(y-at)\right\} \tag{4.5}$$

which is plotted in fig.(12). The Alfvén waves travel in the positive y-direction with zero strength initially. As time advances, the disturbances in both the velocity and magnetic field increase (i.e., the strength of the Alfvén wave increases). The velocity at the interface is one half that of the wall, indicating a slip.

(3) $\epsilon << 1$

The induced magnetic field in the case of vanishingly small ϵ is of order ϵ and the velocity field is

$$\begin{aligned} U(y,t) &= \tfrac{1}{2}U_o \Big\{ \exp(ay/\sqrt{\nu\eta})\,\mathrm{erfc}(y/2\sqrt{\nu t} + a\sqrt{t/\eta}) + \\ &\exp(-ay/\sqrt{\nu\eta})\,\mathrm{erfc}(y/2\sqrt{\nu t} - a\sqrt{t/\eta}) - \exp(-wt)\big[\\ &\exp((a^2 - \eta w)^{\frac{1}{2}} y/\sqrt{\nu\eta})\,\mathrm{erfc}(y/2\sqrt{\nu t} + \sqrt{(a^2 - \eta w)t/\eta}) + \\ &\exp(-(a^2 - \eta w)^{\frac{1}{2}} y/\sqrt{\nu\eta})\,\mathrm{erfc}(y/2\sqrt{\nu t} - \sqrt{(a^2 - \eta w)t/\eta}) \\ &+ O(\epsilon)\Big\} \end{aligned}$$

$$\text{for } w \leq a^2/\eta \, , \tag{4.6}$$

$$\begin{aligned} U(y,t) &= \tfrac{1}{2}U_o \Big\{ \exp(ay/\sqrt{\nu\eta})\,\mathrm{erfc}(y/2\sqrt{\nu t} + a\sqrt{t/\eta}) + \\ &\exp(-ay/\sqrt{\nu\eta})\,\mathrm{erfc}(y/2\sqrt{\nu t} - a\sqrt{t/\eta}) - \\ &2\exp(-wt)\,\mathrm{erfc}(y/2\sqrt{\nu t}) + O(\epsilon)\Big\} \end{aligned}$$

$$\text{for } w = a^2/\eta \, . \tag{4.7}$$

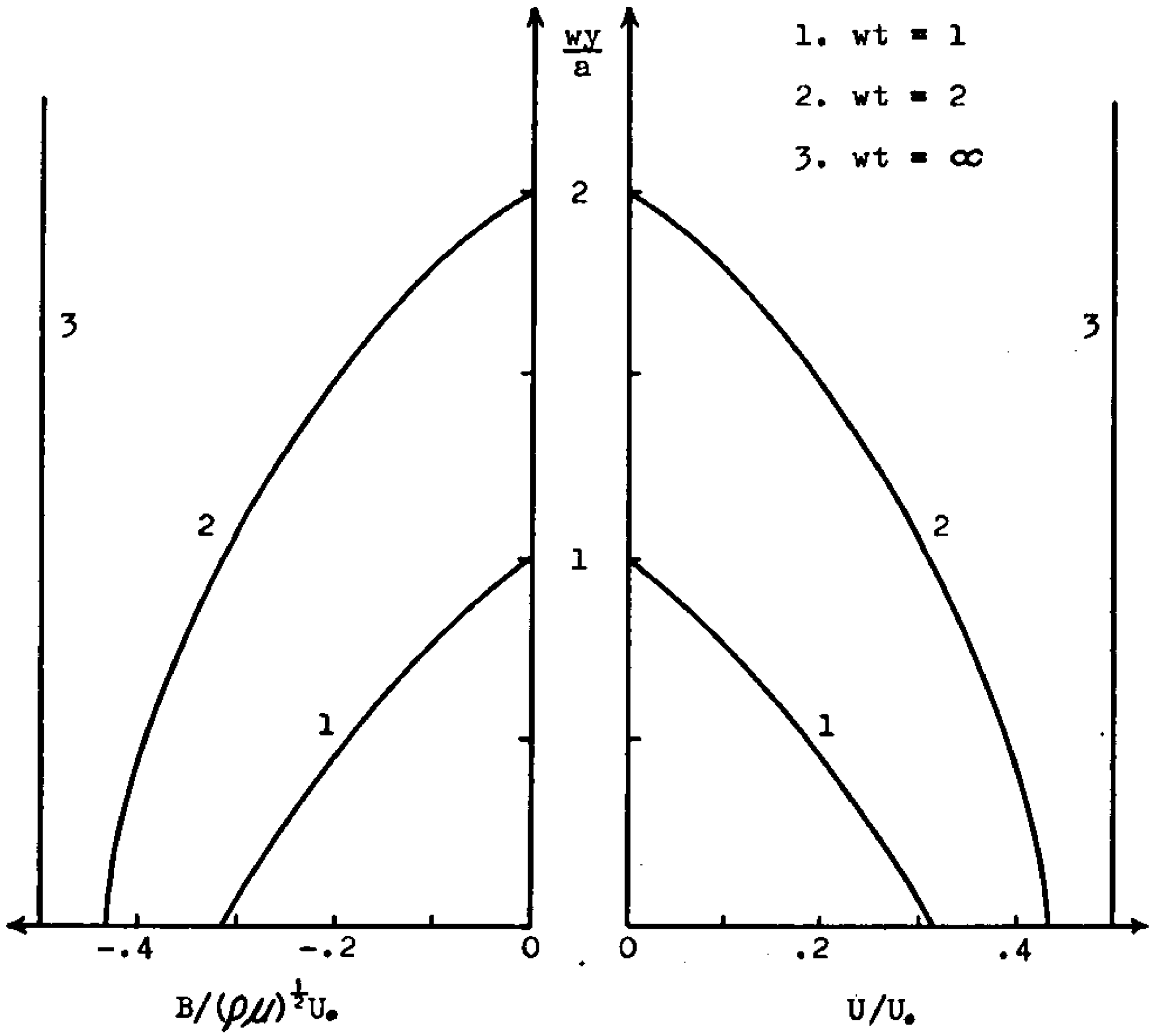

Fig. 12. Velocity and magnetic field profiles: accelerating motion of a non-conducting wall, $\nu = \eta = 0$.

3. Shear stress at the solid-fluid interface

The shear stress at the interface is given by eqs. (3.20)-(3.22) as

$$C_f = \frac{2\nu^{\frac{1}{2}}}{U_*(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})} \Big\{ a \operatorname{erf}\Big(\frac{at^{\frac{1}{2}}}{\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}}}\Big) - \exp(-wt)$$

$$(a^2 - (\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2 w)^{\frac{1}{2}} \operatorname{erf}\Big(\sqrt{(a^2/(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2 - w)t}\Big)$$

$$\text{for } w \leq a^2/(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2 \,, \qquad (4.8)$$

$$C_f = \frac{2a\nu^{\frac{1}{2}}}{U_*(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})} \operatorname{erf}\Big(\frac{at^{\frac{1}{2}}}{\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}}}\Big)$$

$$\text{for } w = a^2/(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2 \,. \qquad (4.9)$$

Eqs.(4.8)-(4.9) are plotted in fig.(13) for $\alpha w/a^2 = 1/4$, 1/8 and 1/16 with $\nu = \eta \neq 0$. The value C_f at the interface increases from zero to a/U_* as time advances, and decreases as the value of the parameter $\alpha w/a^2$ decreases.

B. Perfectly Conducting Wall

1. Boundary conditions at the solid-fluid interface

The boundary conditions at the interface for the accelerating motion of a perfectly conducting wall are given by eqs.(2.9) and (2.21), together with wq.(1.5), as

$$(u)_{\xi = 0_+} = I(\tau)u_*[1 - \exp(-\Omega\tau)] \,,$$

$$\left(\frac{\partial b}{\partial \xi}\right)_{\xi = 0_+} = 0 \,.$$

The induced magnetic field in the solid is zero and the

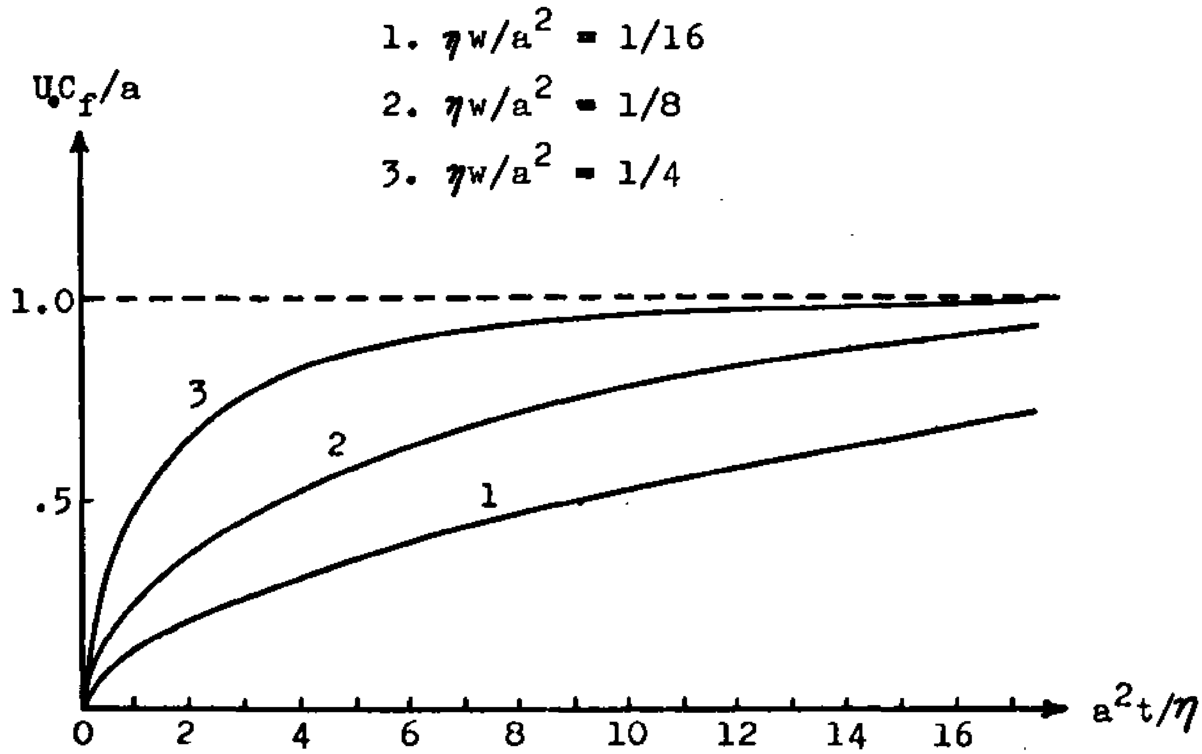

Fig. 13. Stress at the interface: accelerating motion of a non-conducting wall, $\nu = \eta \neq 0$.

solution in the fluid comes directly from those of Chap. III, B by the method of superposition.

2. Solutions in some special cases

$$(1)\ \nu = \eta \neq 0$$

The solution for the case of equal but non-zero diffusion coefficients with $w = a^2/4\eta$ is

$$\left\{\begin{matrix} U(y,t) \\ (\rho\mu)^{-\frac{1}{2}}B(y,t) \end{matrix}\right\}$$

$$= \pm \tfrac{1}{2}U_0\Big\{\operatorname{erfc}\Big(\frac{y - at}{2(\eta t)^{\frac{1}{2}}}\Big) \pm \operatorname{erfc}\Big(\frac{y + at}{2(\eta t)^{\frac{1}{2}}}\Big) - \exp(-wt)$$

$$\Big[\left\{\begin{matrix} \cosh(ay/2\eta) - a(a^2 - 4\eta w)^{-\frac{1}{2}}\sinh(ay/2\eta) \\ \sinh(ay/2\eta) - a(a^2 - 4\eta w)^{-\frac{1}{2}}\cosh(ay/2\eta) \end{matrix}\right\}$$

$$\exp((a^2 - 4\eta w)^{\frac{1}{2}}y/2\eta)\operatorname{erfc}\Big(\frac{y + (a^2 - 4\eta w)^{\frac{1}{2}}t}{2(\eta t)^{\frac{1}{2}}}\Big) +$$

$$\left\{\begin{matrix} \cosh(ay/2\eta) + a(a^2 - 4\eta w)^{-\frac{1}{2}}\sinh(ay/2\eta) \\ \sinh(ay/2\eta) + a(a^2 - 4\eta w)^{-\frac{1}{2}}\cosh(ay/2\eta) \end{matrix}\right\}$$

$$\exp(-(a^2 - 4\eta w)^{\frac{1}{2}}y/2\eta)\operatorname{erfc}\Big(\frac{y - (a^2 - 4\eta w)^{\frac{1}{2}}t}{2(\eta t)^{\frac{1}{2}}}\Big)\Big]\Big\}$$

$$\text{for } w < a^2/4\eta, \qquad (4.10)$$

and

$$\left\{\begin{matrix} U(y,t) \\ (\rho\mu)^{-\frac{1}{2}}B(y,t) \end{matrix}\right\}$$

$$= \pm \tfrac{1}{2}U_0\Big\{\operatorname{erfc}\Big(\frac{y - at}{2(\eta t)^{\frac{1}{2}}}\Big) \pm \operatorname{erfc}\Big(\frac{y + at}{2(\eta t)^{\frac{1}{2}}}\Big) -$$

$$2\exp(-wt)\left[\begin{Bmatrix}\cosh(ay/2\eta) - (ay/2\eta)\sinh(ay/2\eta)\\ \sinh(ay/2\eta) - (ay/2\eta)\cosh(ay/2\eta)\end{Bmatrix}\right.$$

$$\left.\mathrm{erfc}(y/2\sqrt{\eta t}) - (at/\pi\eta)^{\frac{1}{2}}\begin{Bmatrix}\sinh(ay/2\eta)\\ \cosh(ay/2\eta)\end{Bmatrix}\exp(-y^2/4\eta t)\right]\Bigg\}$$

$$\text{for } w = a^2/4\eta \ . \qquad (4.11)$$

Eqs.(4.10)-(4.11) are plotted in fig.(14) for $a^2t/\eta = 1$ and 9 with $\eta w/a^2 = 1/4$ and $1/16$. The disturbances in both the velocity and magnetic field increase with the increase of time and the value of the parameter $\eta w/a^2$ and are larger than those for the non-conducting wall. The induced magnetic field is not negligible except for small time.

(2) $\nu = \eta = 0$

The solution for the case of a perfectly conducting, inviscid, incompressible fluid with $\epsilon = 1$ is

$$\begin{Bmatrix} U(y,t) \\ (\rho\mu)^{-\frac{1}{2}}B(y,t)\end{Bmatrix}$$

$$= \pm\left\{1 - \exp(w(y - at)/a)\right\}\left\{1 - I(y-at)\right\} \qquad (4.12)$$

which is plotted in fig.(15). This is similar to the solution for the non-conducting wall except that there is no slip at the interface and the magnitudes of the velocity and induced magnetic field are twice for the non-conducting wall.

(3) $\epsilon \ll 1$

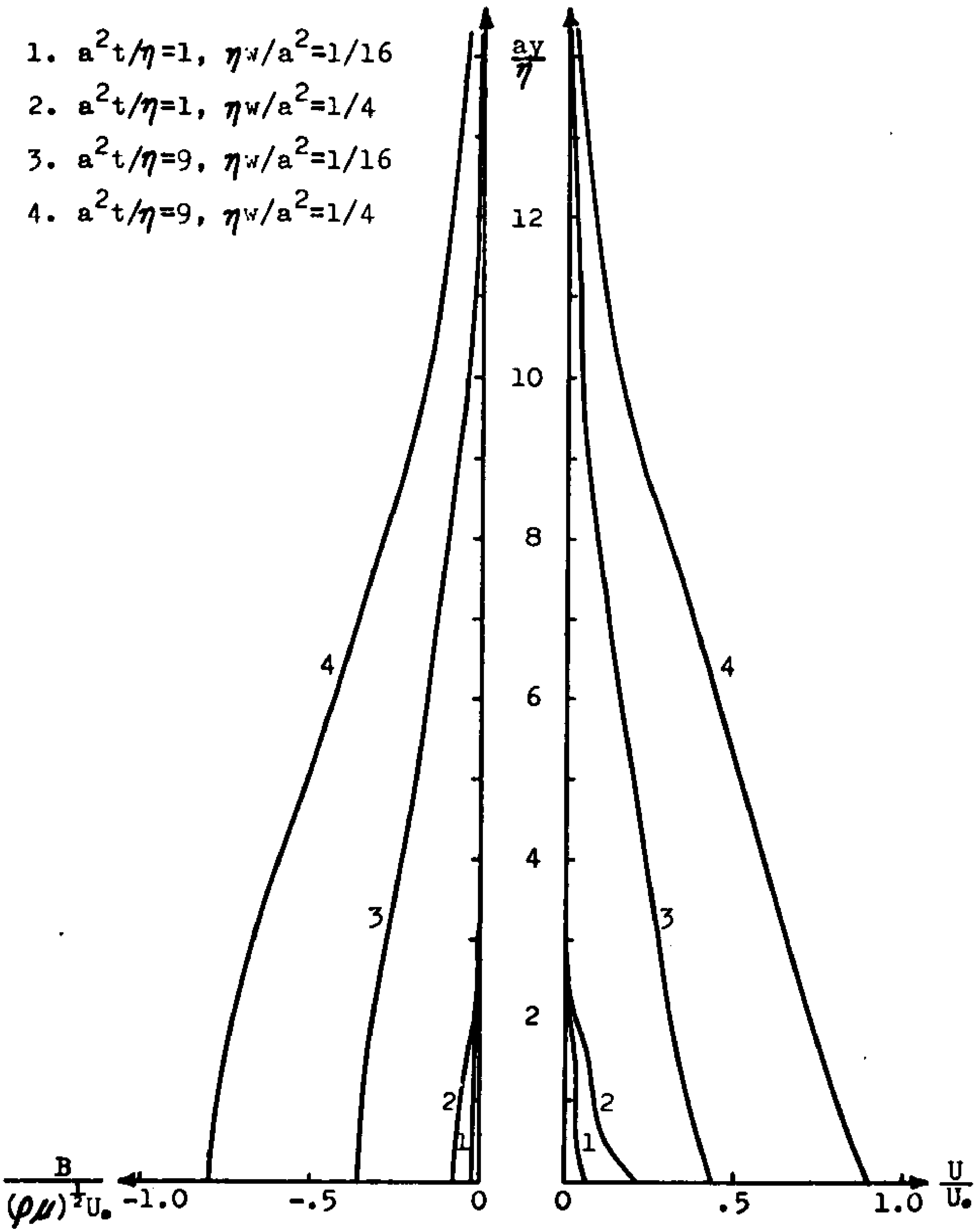

Fig. 14. Velocity and magnetic field profiles: accelerating motion of a perfectly conducting wall, $\nu = \eta \neq 0$.

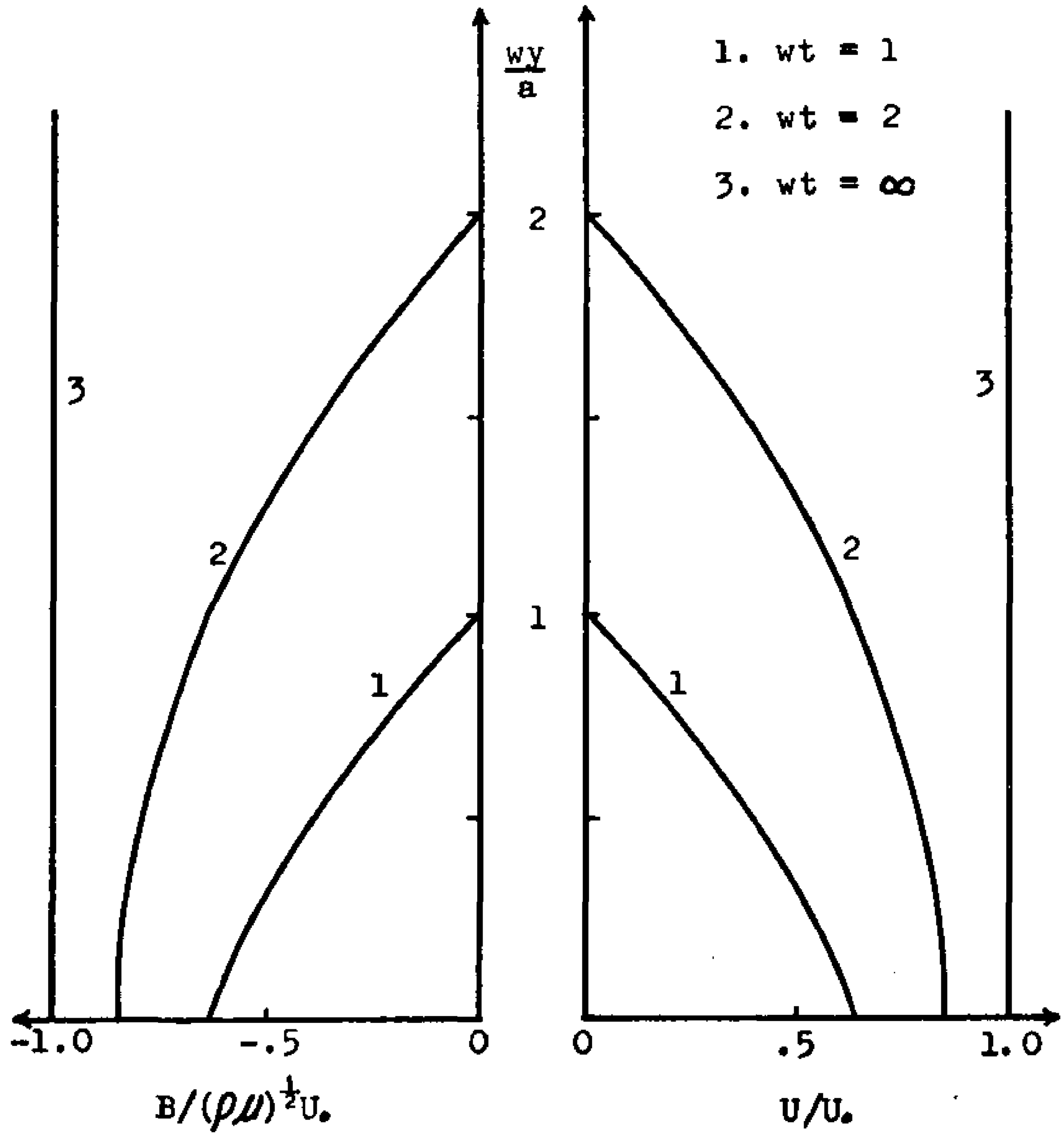

Fig. 15. Velocity and magnetic field profiles: accelerating motion of a perfectly conducting wall, $\nu = \eta = 0$.

The solution for small ay/α with vanishingly small ϵ is

$$U(y,t)/U_0$$
$$= 1 - \exp(-a^2t/\eta)\mathrm{erf}(y/2\sqrt{\nu t}) - \exp(-wt) + \frac{a^2}{a^2 - \eta w}$$
$$\Big\{\exp(-a^2t/\eta)\mathrm{erf}(y/2\sqrt{\nu t}) - (\eta w/2a^2)\exp(-wt)$$
$$\Big[\exp((a^2 - \eta w)^{\frac{1}{2}}y/\sqrt{\nu\eta})\mathrm{erfc}(y/2\sqrt{\nu t} + \sqrt{(a^2 - \eta w)t/\eta}) +$$
$$\exp(-(a^2 - \eta w)^{\frac{1}{2}}y/\sqrt{\nu\eta})\mathrm{erfc}(y/2\sqrt{\nu t} - \sqrt{(a^2 - \eta w)t/\eta})$$
$$-2\Big]\Big\} \qquad \text{foe } w < a^2/\eta\ ,$$
$$U(y,t)/U_0$$
$$= 1 - \exp(-wt) - \exp(-a^2t/\eta)\Big\{(a^2t/\eta)\mathrm{erf}(y/2\sqrt{\nu t}) -$$
$$(a^2y^2/2\nu\eta)\mathrm{erf}(y/2\sqrt{\nu t}) + (4a^2y/\eta)(t/\pi\nu)^{\frac{1}{2}}\exp(-y^2/4\nu t)$$
$$\text{for } w = a^2/\eta\ ;$$

and

$$-B(y,t)/(\rho\mu)^{\frac{1}{2}}U_0$$
$$= \mathrm{erf}(a\sqrt{t/\eta}) - a(a^2 - \eta w)^{-\frac{1}{2}}e^{-wt}\mathrm{erf}(\sqrt{(a^2 - \eta w)t/\eta})$$
$$\text{for } w < a^2/\eta\ ,$$
$$-B(y,t)/(\rho\mu)^{\frac{1}{2}}U_0$$
$$= \mathrm{erf}(a\sqrt{t/\eta}) - 2(a^2t/\pi\eta)\exp(-a^2t/\eta)$$
$$\text{for } w = a^2/\eta\ .$$

Note that the disturbance in the magnetic field is not negligible as it is in the case of a non-conducting wall.

3. Shear stress at the solid-fluid interface

The shear stress at the interface is

$$C_f = \frac{2a}{U_0}\left\{\operatorname{erf}\left(\frac{at^{\frac{1}{2}}}{\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}}}\right) - \frac{a}{(a^2 - (\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2 w)^{\frac{1}{2}}}\left[1 - \nu^{\frac{1}{2}}(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})w/a^2\right] e^{-wt} \operatorname{erf}\left(\sqrt{(a^2/(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2 - w)t}\right)\right\}$$

$$\text{for } w < a^2/(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2 ,$$

$$C_f = \frac{2a}{U_0} \operatorname{erf}\left(\frac{at^{\frac{1}{2}}}{\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}}}\right) - 2a(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^{-2}(\eta t/\pi)^{\frac{1}{2}} \exp\left(- \frac{a^2 t}{(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2}\right)$$

$$\text{for } w = a^2/(\nu^{\frac{1}{2}} + \eta^{\frac{1}{2}})^2 .$$

They are plotted in fig.(16) for $\alpha w/a^2 = 1/4$, $1/8$ and $1/16$ with $\nu = \eta \neq 0$. The stress, C_f, at the interface increases from zero to $2a/U_0$ with the increase of time and the value of the parameter $\alpha w/a^2$.

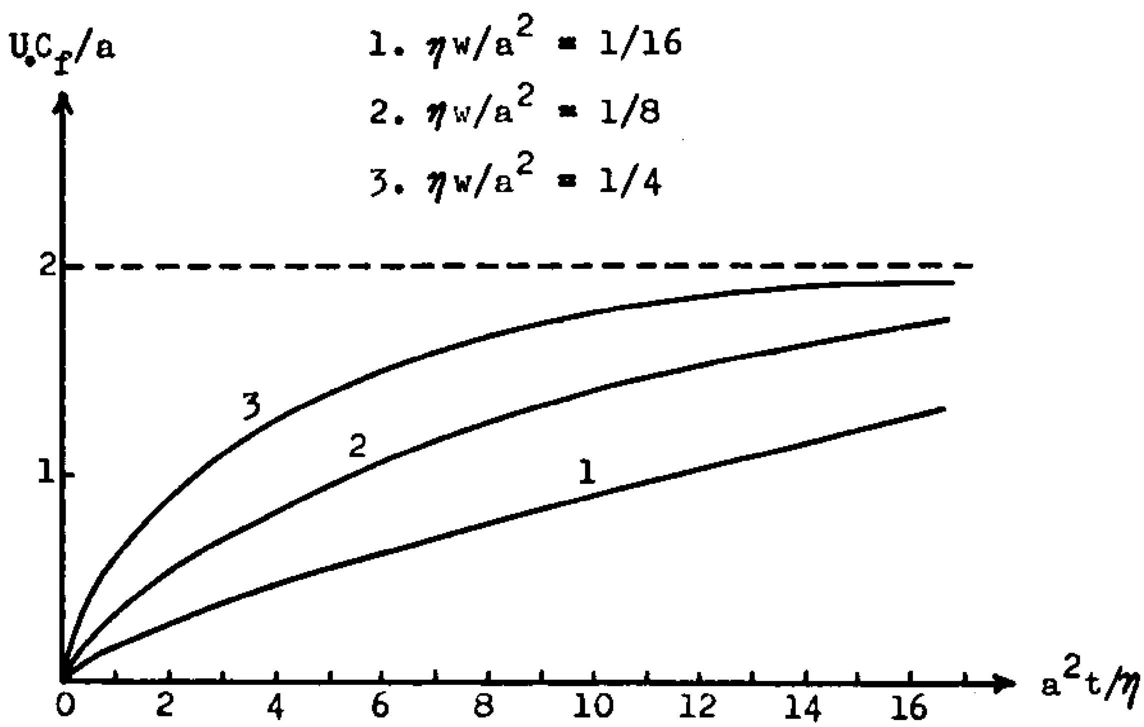

Fig. 16. Stress at the interface: accelerating motion of a perfectly conducting wall, $\nu = \eta \neq 0$.

V. TRANSIENT UNIDIRECTIONAL FLOW OF A NON-DISSIPATIVE FLUID

A. The MHD Rayleigh Problem with a Non-dissipative Fluid

The solutions previously obtained for the decelerating impulsive motion of a perfectly conducting or non-conducting wall with a non-dissipative fluid are only for the case of equal diffusion coefficients, i.e., $\nu = \eta = 0$. Moreover, the solution for the oscillating wall with a non-dissipative fluid can not be obtained as the limiting case of any really transient solution which is available. Here an attempt, in another approach, is made to find the general solution for the transient, unidirectional flow of a non-dissipative fluid with an arbitrary wall motion. The ratio of the diffusion coefficients is arbitrary.

In the MHD Rayleigh problem, as mentioned before, an Alfvén wave is emitted from the solid-fluid interface into the fluid at the Alfvén speed. The Hartmann layer and the range of the diffusing Alfvén wave both shrink to a line in the non-dissipative limit, and the flow behind the wave is quasi-steady. The solutions for the velocity and magnetic intensity are governed, in the case of a non-conducting wall, by eq.(1.3), the Hartmann-Stewartson condition, which can be written as

$$[\underline{B}]_s = \pm (\rho\mu)^{\frac{1}{2}}\epsilon[\underline{V}]_s \qquad (5.1)$$

across the solid-fluid interface, and the Alfvén condition,

$$[\underline{B}]_s = \mp (\rho\mu)^{\frac{1}{2}}[\underline{V}]_s ,$$

across the Alfvén wave. The solutions have been shown by Dix [17]. Dix's method, however, can not be applied to the case of unsteady wall motion. We therefore have to reexamine the governing equations and the solutions obtained before for the decelerating wall motion with $\nu = \eta = 0$.

B. Governing Equations and Boundary Conditions

Eqs.(2.1) and (2.2) with zero diffusion coefficients, ν and η can be written as

$$D_\pm(U \mp \beta B) = 0 \qquad (5.2)$$

where

$$\beta = (\rho\mu)^{-\frac{1}{2}}$$

and

$$D_\pm = \frac{\partial}{\partial t} \pm a \frac{\partial}{\partial y}$$

which is a differentiation in the direction (1, 0, $\pm$a) in (t, x, y) space. Eqs.(5.2) imply that constant signals $U \mp \beta B$ are transmitted along the $l_\pm$ signal lines defined by

$$l_\pm : \quad dt = \frac{dy}{\pm a}$$

i.e., along the positive or negative direction of the applied magnetic field with the Alfvén speed a.

The $l_\pm$ signal lines must have originated either on the solid-fluid interface or in the fluid at $t = 0$ since these signal lines can not terminate in the fluid at any non-zero time. Thus the signals they transmit are determined either by interface data or by undisturbed initial values. On the solid-fluid interface, only one type of signal line can point into the fluid. This is clear since the projection of the $l_{\pm}$ signal lines on the x, y-plane is given by $(0, \pm a)$. In our problem the l_+ signal lines, carrying constant signals $U-\beta B$, are directed from the interface into the fluid. Bundles of these signal lines constitute the wave region. The l_- signal lines, originating in the fluid at $t = 0$, carry undisturbed $U-\beta B$ signals, giving

$$U - \beta B = 0 \qquad (5.4)$$

in the fluid. Outside the wave region, the flow is undisturbed since both $l_{\pm}$ lines carry undisturbed values.

The determination of the constant signals $U-\beta B$, originating at the interface, requires the knowledge of the values U and B at the interface for all time. In the case of a non-conducting wall, eqs.(5.1) and (5.4) are used to to solve for U and B at the interface. In the case of a perfectly conducting wall, the Hartmann-Stewartson condition is replaced by the condition of continuity of the tangential component of electrical field across the interface. This requires that the velocity field is continuous across the interface since the applied magnetic field is a non-zero constant.

C. Decelerating Impulsive Wall Motion with a Non-dissipative Fluid

It is worthwhile to examine how the constant signals $U-\beta B$ are transmitted from the interface into the fluid along l_+ lines in the exact solutions obtained before. The solution for the decelerating, impulsive, non-conducting wall with velocity $U_{wall} = I(t)U_* \exp(-wt)$ can be written as

$$\left\{ \begin{matrix} U(y,t) \\ -\beta B(y,t) \end{matrix} \right\} = \tfrac{1}{2}U_* \exp\{-w(t - y/a)\}\{1 - I(y-at)\} \tag{5.5}$$

in the case $\nu = \eta = 0$. It is clear that eqs.(5.1) and (5.2) are satisfied across the interface and Alfvén wave. Also, eq.(5.4) is satisfied in the fluid. The constant signals transmitted from the interface into the fluid along the l_+ lines are

$$U - \beta B = U_* \exp(-wt')\{1 - I(y-at)\} \tag{5.6}$$

where $t' = t - y/a$. The expression on the right hand side of eq.(5.6) represents the constant signals which reach the position (x,y) at time t and were emitted from the interface at time (t - y/a) as expected. Similar arguments can be applied in the case of a perfectly conducting wall.

D. Solution for Arbitrary Wall Motion with a Non-dissipative Fluid

1. Solution at the interface

In the case of a non-conducting wall with velocity $U_w(t)$, eq.(5.1) reads

$$\epsilon U(0_+,t) - \beta B(0_+,t) = \epsilon U_w \tag{5.7}$$

since the induced magnetic field is zero in the wall. The solution of eqs.(5.7) and (5.4) for U and B at the interface is

$$U(0_+,t) = -\beta B(0_+,t) = \frac{\epsilon}{1+\epsilon} U_w(t) \ . \tag{5.8}$$

This provides the values of the signals $U-\beta B$ on the interface at time t, i.e.,

$$U - \beta B = \frac{2\epsilon}{1+\epsilon} U_w(t) \ . \tag{5.9}$$

These signals will be transmitted into the fluid along the l_+ lines.

In the case of a perfectly conducting wall with velocity $U_w(t)$, the velocity field is continuous across the interface, and eq.(5.4) reads

$$U(0_+,t) = -\beta B(0_+,t) = U_w(t) \ . \tag{5.10}$$

This again provides the values of the signals $U-\beta B$ on the interface at time t;

$$U - \beta B = 2U_w(t) \ , \tag{5.11}$$

which will be transmitted into the fluid along the l_+ lines.

2. Solution for the unsteady unidirectional flow of a non-dissipative fluid

The constant signals, $U-\beta B$, reaching the position (x,y) at time t inside the wave region, are those transmitted from the interface at time $t' = t - y/a$. Replacing t by t' on the right hand side of eq.(5.9), we then have

$$U(y,t) - \beta B(y,t) = \frac{2\epsilon}{1+\epsilon} U_w(t') \tag{5.12}$$

in the wave region for a non-conducting wall. The solution of eqs.(5.12) and (5.4) for U(y,t) and B(y,t) is

$$\left\{ \begin{matrix} U(y,t) \\ -\beta B(y,t) \end{matrix} \right\} = \frac{\epsilon}{1+\epsilon} U_w(t') \left\{ 1 - I(y-at) \right\} \tag{5.13}$$

for a non-conducting wall.

Similarly, the solution for a perfectly conducting wall is

$$\left\{ \begin{matrix} U(y,t) \\ -\beta B(y,t) \end{matrix} \right\} = U_w(t') \left\{ 1 - I(y-at) \right\} \tag{5.14}$$

which can be obtained from eqs.(5.11) and (5.4).

3. Example: solution for an oscillating wall

If the velocity of an oscillating wall is assumed to be $I(t)U_\circ \sin(wt)$, the solution for U(y,t) and B(y,t) is

$$\left\{ \begin{matrix} U(y,t) \\ -\beta B(y,t) \end{matrix} \right\} = \frac{\epsilon}{1+\epsilon} U_\circ \sin\{w(t - y/a)\} \left\{ 1 - I(y-at) \right\} \tag{5.15}$$

for a non-conducting wall and

$$\left\{ \begin{matrix} U(y,t) \\ -\beta B(y,t) \end{matrix} \right\} = U_\circ \sin\{w(t - y/a)\} \left\{ 1 - I(y-at) \right\} \tag{5.16}$$

for a perfectly conducting wall. Eqs.(5.15) with $\epsilon = 1$ are plotted in fig.(17).

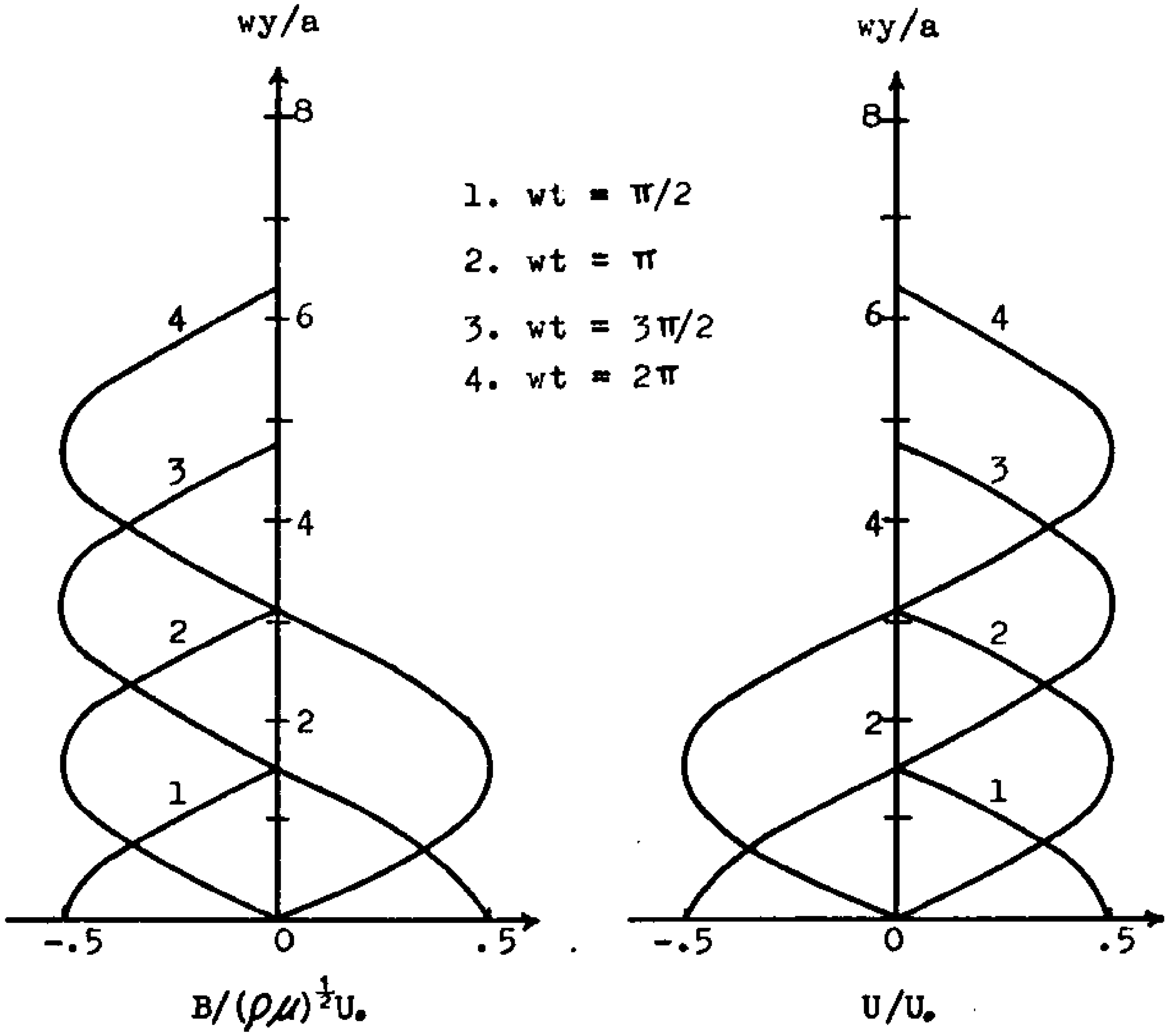

Fig. 17. Velocity and magnetic field profiles: oscillating wall motion ($U_{wall} = U_0 \sin(wt)$), $\nu = \eta = 0$.

VI. APPENDIX

The following Laplace transforms are encountered in eqs.(3.6), (3.12), (3.28) and (3.38):

$$(1/s)\exp(-(s + c)^{\frac{1}{2}}k), \quad (1/s)\left(\frac{c}{s + c}\right)^{\frac{1}{2}}\exp(-(s - c)^{\frac{1}{2}}k)$$

and $\quad (1/s^2)\exp(-s^{\frac{1}{2}}k)$

which can not be found in tables [18]. Here s is the transform variable, c and k are positive constants. We wish to show that

$$(1)\ L^{-1}\left\{(1/s)\exp(-(s + c)^{\frac{1}{2}}k)\right\}$$

$$= \tfrac{1}{2}\left\{\exp(c^{\frac{1}{2}}k)\operatorname{erfc}\left(\frac{k + 2c^{\frac{1}{2}}t}{2t^{\frac{1}{2}}}\right) + \right.$$

$$\left.\exp(-c^{\frac{1}{2}}k)\operatorname{erfc}\left(\frac{k - 2c^{\frac{1}{2}}t}{2t^{\frac{1}{2}}}\right)\right\}, \tag{6.1}$$

$$(2)\ L^{-1}\left\{(1/s)\left(\frac{c}{s + c}\right)^{\frac{1}{2}}\exp(-(s + c)^{\frac{1}{2}}k)\right\}$$

$$= \tfrac{1}{2}\left\{\exp(-c^{\frac{1}{2}}k)\operatorname{erfc}\left(\frac{k - 2c^{\frac{1}{2}}t}{2t^{\frac{1}{2}}}\right) - \right.$$

$$\left.\exp(c^{\frac{1}{2}}k)\operatorname{erfc}\left(\frac{k + 2c^{\frac{1}{2}}t}{2t^{\frac{1}{2}}}\right)\right\}, \tag{6.2}$$

$$(3)\ L^{-1}\ (1/s^2)\exp(-s^{\frac{1}{2}}k)$$

$$= (t + \tfrac{1}{2}k^2)\operatorname{erfc}(k/2t^{\frac{1}{2}}) - k(t/\pi)^{\frac{1}{2}}\exp(-k^2/4t) \tag{6.3}$$

where

$$L^{-1}\{\bar{u}(s)\} = u(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}u(s)ds \ .$$

To derive eq.(5.1), we note that

$$L^{-1}\{\exp(-(s + c)^{\frac{1}{2}}k)\}$$

$$= \exp(-ct)\ L^{-1}\{\exp(-s^{\frac{1}{2}}k)\}$$

$$= \frac{k}{2(\pi t^3)^{\frac{1}{2}}} \exp(-ct - k^2/4t) \ .$$

Then, by convolution

$$L^{-1}\{(1/s)\exp(-(s + c)^{\frac{1}{2}}k)\}$$

$$= \int_0^t \frac{k}{2(\pi t^3)^{\frac{1}{2}}} \exp(-ct - k^2/4t)dt$$

$$= \exp(c^{\frac{1}{2}}k) \int_0^t \frac{k - 2c^{\frac{1}{2}}t}{4(\pi t^3)^{\frac{1}{2}}} \exp(-(\frac{k + 2c^{\frac{1}{2}}t}{2t^{\frac{1}{2}}})^2)dt +$$

$$\exp(-c^{\frac{1}{2}}k) \int_0^t \frac{k + 2c^{\frac{1}{2}}t}{4(\pi t^3)^{\frac{1}{2}}} \exp(-(\frac{k - 2c^{\frac{1}{2}}t}{2t^{\frac{1}{2}}})^2)dt$$

$$= \tfrac{1}{2}\exp(c^{\frac{1}{2}}k)\frac{2}{\pi^{\frac{1}{2}}} \int_{\frac{k+2c^{\frac{1}{2}}t}{2t^{\frac{1}{2}}}}^{\infty} \exp(-x^2)dx +$$

$$\tfrac{1}{2}\exp(-c^{\frac{1}{2}}k)\frac{2}{\pi^{\frac{1}{2}}} \int_{\frac{k-2c^{\frac{1}{2}}t}{2t^{\frac{1}{2}}}}^{\infty} \exp(-x^2)dx$$

$$= \tfrac{1}{2}\exp(c^{\frac{1}{2}}k)\operatorname{erfc}(\frac{k + 2c^{\frac{1}{2}}t}{2t^{\frac{1}{2}}}) + \tfrac{1}{2}\exp(-c^{\frac{1}{2}}k)\operatorname{erfc}(\frac{k - 2c^{\frac{1}{2}}t}{2t^{\frac{1}{2}}})$$

as given by eq.(6.1). Eq.(6.2) is derived similarly. To prove eq.(6.3), we note that

$$L^{-1}\left\{(1/s)\exp(-s^{\frac{1}{2}}k)\right\} = \operatorname{erfc}(k/2\sqrt{t}) \;, \tag{6.4}$$

then

$$L^{-1}\left\{(1/s^2)\exp(-s^{\frac{1}{2}}k)\right\} = \int_0^t \operatorname{erfc}(k/2\sqrt{t})dt$$

by convolution. Since

$$\frac{d}{dt}(t \operatorname{erfc}(k/2\sqrt{t})) = \operatorname{erfc}(k/2\sqrt{t}) + (k/2\sqrt{\pi t}) \exp(-k^2/4t^2) \;,$$

we have

$$\int_0^t \operatorname{erfc}(k/2\sqrt{t})dt$$

$$= t \operatorname{erfc}(k/2\sqrt{t}) - \frac{2}{\pi^{\frac{1}{2}}} \int_0^t \frac{k}{4t^{\frac{1}{2}}} \exp(-k^2/4t)dt \;. \tag{6.5}$$

Now let

$$u = \exp(-k^2/4t) \;, \qquad dv = k/4t^{\frac{1}{2}} \;;$$

then it is easy to show that

$$du = (k^2/4t^2) \exp(-k^2/4t) \;, \qquad v = \tfrac{1}{2}kt^{\frac{1}{2}} \;.$$

Integrating by parts, the second term on the right hand side of eq.(6.5) becomes

$$\frac{2}{\pi^{\frac{1}{2}}} \int_0^t (k/4t^{\frac{1}{2}})\exp(-k^2/4t)dt$$

$$= \frac{kt^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \exp(-k^2/4t) - \frac{k^2}{\pi^{\frac{1}{2}}} \int_0^t (k/4t^{3/2})\exp(-k^2/4t)dt$$

$$= \frac{kt^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \exp(-k^2/4t) - \frac{k^2}{\pi^{\frac{1}{2}}} \int_{\frac{k}{2t^{\frac{1}{2}}}}^{\infty} \exp(-t^2)dt$$

$$= \frac{kt^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \exp(-k^2/4t) - \tfrac{1}{2}k^2 \operatorname{erfc}(k/2\sqrt{t}) \ .$$

Substituting into eq.(6.5), together with eq.(6.4), leads to eq.(6.3) which completes the proof.

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Vitae

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Title of Thesis: The M. H. D. Rayleigh Problem with Unsteady Wall Motion

Approved:

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

Date of Examination:

December 16, 1969